TIGHT BINDING BOOK

EIGENFUNCTION EXPANSIONS
ASSOCIATED WITH
SECOND-ORDER DIFFERENTIAL EQUATIONS

BY

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PREFACE

THE idea of expanding an arbitrary function in terms of the solutions of a second-order differential equation goes back to the time of Sturm and Liouville, more than a hundred years ago. The first satisfactory proofs were constructed by various authors early in the twentieth century. Later, a general theory of the 'singular' cases was given by Weyl, who based it on the theory of integral equations. An alternative method, proceeding via the general theory of linear operators in Hilbert space, is to be found in the treatise by Stone on this subject.

Here I have adopted still another method. Proofs of these expansions by means of contour integration and the calculus of residues were given by Cauchy, and this method has been used by several authors in the ordinary Sturm–Liouville case. It is applied here to the general singular case. It is thus possible to avoid both the theory of integral equations and the general theory of linear operators, though of course we are sometimes doing no more than adapt the latter theory to the particular case considered.

The ordinary Sturm–Liouville expansion is now well known. I therefore dismiss it as rapidly as possible, and concentrate on the 'singular' cases, a class which seems to include all the most interesting examples. In order to present a clear-cut theory in a reasonable space, I have had to reject firmly all generalizations. Many of the arguments used extend quite easily to other cases, such as that of two simultaneous first-order equations.

It seems that physicists are interested in some aspects of these questions. If any physicist finds here anything that he wishes to know, I shall indeed be delighted; but it is to mathematicians that the book is addressed. I believe in the future of 'mathematics for physicists', but it seems desirable that a writer on this subject should understand physics as well as mathematics.

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I

THE STURM–LIOUVILLE EXPANSION

1.1. Introduction. Let L denote a linear operator operating on a function $y = y(x)$. Consider the equation

$$Ly = \lambda y, \tag{1.1.1}$$

where λ is a number. A function which satisfies this equation and also certain boundary conditions (e.g. which vanishes at $x = a$ and $x = b$) is called an eigenfunction. The corresponding value of λ is called an eigenvalue. Thus if $\psi_n(x)$ is an eigenfunction corresponding to an eigenvalue λ_n,

$$L\psi_n(x) = \lambda_n \psi_n(x). \tag{1.1.2}$$

The object of this book is to study the operator

$$L \equiv q(x) - \frac{d^2}{dx^2}, \tag{1.1.3}$$

where $q(x)$ is a given function of x defined over some given interval (a, b). In this case y satisfies the second-order differential equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0, \tag{1.1.4}$$

and $\psi_n(x)$ satisfies

$$\psi_n''(x) + \{\lambda_n - q(x)\}\psi_n(x) = 0. \tag{1.1.5}$$

If we take this and the corresponding equation with m instead of n, multiply by $\psi_m(x)$, $\psi_n(x)$ respectively, and subtract, we obtain

$$\begin{aligned}(\lambda_m - \lambda_n)\psi_m(x)\psi_n(x) &= \psi_m(x)\psi_n''(x) - \psi_n(x)\psi_m''(x) \\ &= \frac{d}{dx}\{\psi_m(x)\psi_n'(x) - \psi_n(x)\psi_m'(x)\}.\end{aligned}$$

Hence

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b \psi_m(x)\psi_n(x)\,dx &= [\psi_m(x)\psi_n'(x) - \psi_n(x)\psi_m'(x)]_a^b \\ &= 0\end{aligned}$$

if $\psi_m(x)$ and $\psi_n(x)$ both vanish at $x = a$ and $x = b$ (or satisfy a more general condition of the same kind). If $\lambda_m \neq \lambda_n$, it follows that

$$\int_a^b \psi_m(x)\psi_n(x)\,dx = 0. \tag{1.1.6}$$

By multiplying if necessary by a constant we can arrange that

$$\int_a^b \psi_n^2(x)\,dx = 1. \tag{1.1.7}$$

The functions $\psi_n(x)$ then form a normal orthogonal set.

Our main problem is to determine under what conditions an arbitrary function $f(x)$ can be expanded in terms of such functions, in the manner of an ordinary Fourier series. If this is possible, and the expansion is

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{1.1.8}$$

then on multiplying by $\psi_m(x)$ and integrating over (a, b), we obtain formally

$$c_m = \int_a^b f(x)\psi_m(x)\,dx. \tag{1.1.9}$$

In some cases the eigenvalues are not discrete points, but form a continuous range, say, for example, over $(0, \infty)$. The expansion then takes the form

$$f(x) = \int_0^{\infty} c(\lambda)\psi_\lambda(x)\,d\lambda. \tag{1.1.10}$$

All this has its simplest illustration in the case of ordinary Fourier series. Suppose, for example, that $q(x) = 0$, and that the interval considered is $(0, \pi)$. The solution of (1.1.4) which vanishes at $x = 0$ is then $y = \sin(x\sqrt{\lambda})$. This vanishes at $x = \pi$ if and only if $\lambda = n^2$, where n is an integer. These then are the eigenvalues, and the corresponding eigenfunctions are the functions $\sin nx$. That an arbitrary function can be expanded in terms of these functions is the familiar theorem on Fourier's sine series.

1.2. An argument which has sometimes been used to suggest the validity of the above expansions runs as follows. Consider the partial differential equation

$$Lf = i\frac{\partial f}{\partial t}, \tag{1.2.1}$$

where $f = f(x, t)$. If $f(x, t)$ is given for one value of t, it is fixed by this equation for a slightly greater value of t. Thus we should expect to have one solution, and only one, for any given initial value of

$f(x,t)$, i.e. for $f(x,t)$ equal to an arbitrary $f(x)$ when $t = 0$. Suppose now that the solution can be expressed as a Fourier integral,

$$f(x,t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(x,\lambda)e^{-i\lambda t}\,d\lambda, \tag{1.2.2}$$

where
$$F(x,\lambda) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x,t)e^{i\lambda t}\,dt. \tag{1.2.3}$$

Substituting in (1.2.1), we obtain

$$\int_{-\infty}^{\infty} e^{-i\lambda t} LF(x,\lambda)\,d\lambda = \int_{-\infty}^{\infty} \lambda e^{-i\lambda t} F(x,\lambda)\,d\lambda.$$

Since this holds for all values of t, we can equate the coefficients of $e^{-i\lambda t}$, which gives

$$LF(x,\lambda) = \lambda\, F(x,\lambda).$$

Thus $F(x,\lambda)$ is an eigenfunction of the operator L belonging to the eigenvalue λ. Now putting $t = 0$ in (1.2.2), we obtain

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(x,\lambda)\,d\lambda,$$

which gives an expression for the arbitrary function $f(x)$ in terms of eigenfunctions. If $f(x,t)$ were expressible in a Fourier series instead of an integral, we should obtain similarly a series expansion.

The difficulty of justifying directly an argument such as the above is obvious.

1.3. The argument assumes, for one thing, that $f(x, t)$ is small as $t \to \infty$, since this is required for the Fourier integral formula (1.2.2), (1.2.3) to hold. However, a more general form of the formula is as follows. Let

$$F_+(x,\lambda) = \frac{1}{\sqrt{(2\pi)}} \int_{0}^{\infty} f(x,t)e^{i\lambda t}\,dt \quad (\mathbf{I}(\lambda) > 0), \tag{1.3.1}$$

$$F_-(x,\lambda) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{0} f(x,t)e^{i\lambda t}\,dt \quad (\mathbf{I}(\lambda) < 0). \tag{1.3.2}$$

The inverse formula is then

$$f(x,t) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{ic-\infty}^{ic+\infty} F_+(x,\lambda)e^{-i\lambda t}\,d\lambda + \frac{1}{\sqrt{(2\pi)}} \int\limits_{ic'-\infty}^{ic'+\infty} F_-(x,\lambda)e^{-i\lambda t}\,d\lambda, \tag{1.3.3}$$

where $c > 0$, $c' < 0$. Using this and (1.2.1), we obtain formally

$$\begin{aligned} LF_+(x,\lambda) &= \frac{1}{\sqrt{(2\pi)}} \int\limits_0^\infty Lf(x,t)e^{i\lambda t}\,dt \\ &= \frac{i}{\sqrt{(2\pi)}} \int\limits_0^\infty \frac{\partial f}{\partial t} e^{i\lambda t}\,dt \\ &= \frac{i}{\sqrt{(2\pi)}} [f(x,t)e^{i\lambda t}]_0^\infty + \frac{\lambda}{\sqrt{(2\pi)}} \int\limits_0^\infty f(x,t)e^{i\lambda t}\,dt \\ &= -\frac{i}{\sqrt{(2\pi)}} f(x) + \lambda F_+(x,\lambda) \end{aligned} \tag{1.3.4}$$

if $f(x,t)$ reduces to $f(x)$ when $t = 0$. Similarly

$$LF_-(x,\lambda) = \frac{i}{\sqrt{(2\pi)}} f(x) + \lambda F_-(x,\lambda). \tag{1.3.5}$$

The method to be employed is therefore as follows. We construct the solutions F_+ and F_- of (1.3.4) and (1.3.5) which satisfy given boundary conditions. Then (1.3.3), with $t = 0$, gives

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{ic-\infty}^{ic+\infty} F_+(x,\lambda)\,d\lambda + \frac{1}{\sqrt{(2\pi)}} \int\limits_{ic'-\infty}^{ic'+\infty} F_-(x,\lambda)\,d\lambda. \tag{1.3.6}$$

In the simplest cases $F_-(x,\lambda)$ is found to be minus the analytic continuation of $F_+(x,\lambda)$ across the real axis. Writing

$$\Phi(x,\lambda) = -i\sqrt{(2\pi)}F_+(x,\lambda) \tag{1.3.7}$$

(1.3.4) becomes

$$(L-\lambda)\Phi(x,\lambda) = -f(x), \tag{1.3.8}$$

and (1.3.6) becomes

$$f(x) = \frac{1}{2\pi i}\left(\int\limits_{ic'-\infty}^{ic'+\infty} + \int\limits_{ic+\infty}^{ic-\infty} \right)\Phi(x,\lambda)\,d\lambda. \tag{1.3.9}$$

The expansion is then obtained from the calculus of residues, the terms of the series being the residues at the poles of $\Phi(x,\lambda)$.

In any case, since (1.3.4) and (1.3.5) differ only in having the sign of i changed, the two terms in (1.3.6) are conjugate. Hence we also have

$$f(x) = \mathbf{R}\left\{\sqrt{\left(\frac{2}{\pi}\right)} \int\limits_{ic-\infty}^{ic+\infty} F_{+}(x,\lambda)\, d\lambda\right\}$$

$$= -\mathbf{I}\left\{\frac{1}{\pi} \int\limits_{ic-\infty}^{ic+\infty} \Phi(x,\lambda)\, d\lambda\right\}. \tag{1.3.10}$$

The expansion formula is obtained from this by making $c \to 0$.

The above argument indicates in particular that λ must be treated as a complex variable; but the analysis is of course still purely formal.

1.4. In the particular case in which the operator L is given by (1.1.3), (1.3.8) is the second-order differential equation

$$\Phi''(x)+\{\lambda-q(x)\}\Phi(x) = f(x). \tag{1.4.1}$$

In this case the function $\Phi(x,\lambda)$ can be expressed in terms of the solutions of (1.1.4). Let $W_x(\phi,\psi)$ or $W(\phi,\psi)$ denote the Wronskian $W(\phi,\psi) = \phi(x)\psi'(x)-\phi'(x)\psi(x)$ of two functions ϕ and ψ. Now let $\phi(x,\lambda)$ and $\psi(x,\lambda)$ be two solutions of (1.1.4) such that $W(\phi,\psi) = 1$. Then a solution of (1.4.1) is

$$\Phi(x,\lambda) = \psi(x,\lambda)\int\limits_a^x \phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int\limits_x^b \psi(y,\lambda)f(y)\,dy, \tag{1.4.2}$$

as is easily verified by differentiating twice.

Another starting-point for the theory, which (in the case of (1.1.3)) avoids an appeal to Fourier integrals, is as follows. Suppose the theory already established, and consider the properties of the function

$$\Phi(x) = \Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n \psi_n(x)}{\lambda-\lambda_n}. \tag{1.4.3}$$

This gives

$$\Phi''(x)-q(x)\Phi(x) = \sum_{n=0}^{\infty} \frac{c_n\{\psi_n''(x)-q(x)\psi_n(x)\}}{\lambda-\lambda_n}$$

$$= \sum_{n=0}^{\infty} \frac{-c_n\lambda_n\psi_n(x)}{\lambda-\lambda_n} = \sum_{n=0}^{\infty} c_n\psi_n(x)\left(1-\frac{\lambda}{\lambda-\lambda_n}\right)$$

$$= f(x)-\lambda\Phi(x).$$

This is (1.4.1) again, so that we are again led to consider solutions of this differential equation. If we can solve it, (1.4.3) then indicates that the terms of the expansion of $f(x)$ will be the residues at the poles of $\Phi(x,\lambda)$.

Our general method consists of defining $\Phi(x,\lambda)$ by (1.4.2); integration round a large contour in the complex λ-plane then gives the value $f(x)$, and the singularities of $\Phi(x,\lambda)$ on the real axis give a series or an integral expansion as the case may be.

1.5. The Sturm–Liouville expansion. We shall suppose throughout that $q(x)$ is a real function of x, continuous at all points interior to the interval (a,b) considered. In the classical Sturm–Liouville case (a,b) is a finite interval, and $q(x)$ also tends to finite limits as $x \to a$ and $x \to b$.

The general theorem on the existence of solutions of (1.1.4) is as follows.

THEOREM 1.5. *If $q(x)$ satisfies the above conditions, and α is given, the equation* (1.1.4) *has a solution $\phi(x)$ $(a \leqslant x \leqslant b)$ such that*

$$\phi(a) = \sin\alpha, \qquad \phi'(a) = -\cos\alpha.$$

For each x, $\phi(x)$ is an integral function of λ.

Let
$$y_0(x) = \sin\alpha-(x-a)\cos\alpha,$$
and for $n = 1, 2,\ldots,$

$$y_n(x) = y_0(x)+\int_a^x \{q(t)-\lambda\}y_{n-1}(t)(x-t)\,dt.$$

Let $|q(x)| \leqslant M$, $|y_0(x)| \leqslant K$, for $a \leqslant x \leqslant b$, and let $|\lambda| \leqslant N$. Then

$$|y_1(x)-y_0(x)| \leqslant \int_a^x (M+N)K(x-t)\,dt = \tfrac{1}{2}(M+N)K(x-a)^2.$$

For $n \geqslant 1$,

$$y_n(x)-y_{n-1}(x) = \int_a^x \{q(t)-\lambda\}\{y_{n-1}(t)-y_{n-2}(t)\}(x-t)\,dt,$$

$$|y_n(x)-y_{n-1}(x)| \leqslant (M+N)(b-a)\int_a^x |y_{n-1}(t)-y_{n-2}(t)|\,dt.$$

Hence
$$|y_2(x)-y_1(x)| \leqslant \frac{K(M+N)^2(b-a)}{2}\int_a^x (t-a)^2\,dt$$

$$= \frac{K(M+N)^2(b-a)(x-a)^3}{3!},$$

and so generally

$$|y_n(x)-y_{n-1}(x)| \leqslant \frac{K(M+N)^n(b-a)^{n-1}(x-a)^{n+1}}{+1)!}.$$

Hence the series

$$\phi(x) = y_0(x)+\sum_{n=1}^{\infty} \{y_n(x)-y_{n-1}(x)\}$$

converges, uniformly with respect to λ if $|\lambda| \leqslant N$, and with respect to x over $a \leqslant x \leqslant b$. Since for $n \geqslant 2$

$$y_n'(x)-y_{n-1}'(x) = \int_a^x \{q(t)-\lambda\}\{y_{n-1}(t)-y_{n-2}(t)\}\, dt,$$

$$y_n''(x)-y_{n-1}''(x) = \{q(x)-\lambda\}\{y_{n-1}(x)-y_{n-2}(x)\},$$

the first and second differentiated series also converge uniformly with respect to x. Hence

$$\begin{aligned}\phi''(x) &= \sum_{n=1}^{\infty} \{y_n''(x)-y_{n-1}''(x)\} \\ &= \{q(x)-\lambda\}\Big[y_0(x)+\sum_{n=2}^{\infty} \{y_{n-1}(x)-y_{n-2}(x)\}\Big] \\ &= \{q(x)-\lambda\}\phi(x),\end{aligned}$$

so that $\phi(x)$ satisfies (1.1.4). It also clearly satisfies the boundary conditions.

1.6. Now let $\phi(x,\lambda)$, $\chi(x,\lambda)$ be the solutions of (1.1.4) such that

$$\left.\begin{aligned}\phi(a,\lambda) &= \sin\alpha, \qquad & \phi'(a,\lambda) &= -\cos\alpha, \\ \chi(b,\lambda) &= \sin\beta, \qquad & \chi'(b,\lambda) &= -\cos\beta.\end{aligned}\right\} \tag{1.6.1}$$

Then

$$\begin{aligned}\frac{d}{dx} W(\phi,\chi) &= \phi(x)\chi''(x)-\chi(x)\phi''(x) \\ &= \{q(x)-\lambda\}\phi(x)\chi(x)-\{q(x)-\lambda\}\chi(x)\phi(x) \\ &= 0.\end{aligned}$$

Hence $W(\phi,\chi)$ is independent of x, and so is a function of λ only say $\omega(\lambda)$. It is clear from the above theorem that it is an integra function of λ.

Let

$$\Phi(x,\lambda) = \frac{\chi(x,\lambda)}{\omega(\lambda)}\int_a^x \phi(y,\lambda)f(y)\, dy+\frac{\phi(x,\lambda)}{\omega(\lambda)}\int_x^b \chi(y,\lambda)f(y)\, dy. \tag{1.6.2}$$

It is at once verified by differentiation that $\Phi(x,\lambda)$ satisfies (1.4.1), and also the boundary conditions

$$\left.\begin{aligned} \Phi(a,\lambda)\cos\alpha+\Phi'(a,\lambda)\sin\alpha &= 0,\\ \Phi(b,\lambda)\cos\beta+\Phi'(b,\lambda)\sin\beta &= 0, \end{aligned}\right\} \tag{1.6.3}$$

for all values of λ.

Suppose that the only zeros of $\omega(\lambda)$ are simple zeros λ_0, λ_1,... on the real axis. Then the Wronskian of $\phi(x,\lambda_n)$ and $\chi(x,\lambda_n)$ is zero, so that $\chi(x,\lambda_n)$ is a constant multiple of $\phi(x,\lambda_n)$, say

$$\chi(x,\lambda_n) = k_n\,\phi(x,\lambda_n). \tag{1.6.4}$$

It follows from the boundary conditions that k_n is neither 0 nor ∞. Hence $\Phi(x,\lambda)$ has the residue

$$\frac{k_n}{\omega'(\lambda_n)}\phi(x,\lambda_n)\int\limits_a^b \phi(y,\lambda_n)f(y)\,dy$$

at $\lambda = \lambda_n$. The above formalities therefore indicate that there should be an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{k_n}{\omega'(\lambda_n)}\phi(x,\lambda_n)\int\limits_a^b \phi(y,\lambda_n)f(y)\,dy. \tag{1.6.5}$$

This is the Sturm–Liouville expansion.

If we start with any two independent solutions $\phi_0(x,\lambda)$ and $\chi_0(x,\lambda)$ of (1.1.4), and write $\omega_0(\lambda) = W(\phi_0,\chi_0)$, then

$$\phi(x,\lambda) = \frac{\phi_0(x)\{\chi_0(a)\cos\alpha+\chi_0'(a)\sin\alpha\}-\chi_0(x)\{\phi_0(a)\cos\alpha+\phi_0'(a)\sin\alpha\}}{\omega_0(\lambda)}$$

and $\chi(x,\lambda)$ is obtained by replacing a, α by b, β. Then

$$\frac{\phi_0(b,\lambda_n)\cos\beta+\phi_0'(b,\lambda_n)\sin\beta}{\phi_0(a,\lambda_n)\cos\alpha+\phi_0'(a,\lambda_n)\sin\alpha} = \frac{\chi_0(b,\lambda_n)\cos\beta+\chi_0'(b,\lambda_n)\sin\beta}{\chi_0(a,\lambda_n)\sin\alpha+\chi_0'(a,\lambda_n)\sin\alpha} = k_n,$$

and the analysis proceeds as before.

The case of ordinary Fourier series is obtained by taking $q(x) = 0$. The equation (1.1.4) is then

$$\frac{d^2y}{dx^2}+\lambda y = 0.$$

Solutions are $\phi_0(x,\lambda) = \cos x\surd\lambda$, $\chi_0(x,\lambda) = \sin x\surd\lambda$, and $\omega_0(\lambda) = \surd\lambda$.

Take first the case $\alpha = 0$, $\beta = 0$. Then

$$\phi(x,\lambda) = -\frac{\sin\{(x-a)\sqrt{\lambda}\}}{\sqrt{\lambda}}, \qquad \chi(x,\lambda) = \frac{\sin\{(b-x)\sqrt{\lambda}\}}{\sqrt{\lambda}},$$

$$\omega(\lambda) = \frac{\sin\{(b-a)\sqrt{\lambda}\}}{\sqrt{\lambda}}.$$

The zeros of $\omega(\lambda)$ are $\lambda_n = \{n\pi/(b-a)\}^2$ $(n = 1, 2,\ldots)$. We have

$$\omega'(\lambda_n) = \frac{b-a}{2\lambda_n}\cos\{(b-a)\sqrt{\lambda_n}\} = \frac{(-1)^n(b-a)}{2\lambda_n},$$

and
$$k_n = \frac{\phi_0(b,\lambda_n)}{\phi_0(a,\lambda_n)} = \frac{\cos\{bn\pi/(b-a)\}}{\cos\{an\pi/(b-a)\}} = (-1)^n.$$

Hence (1.6.5) gives Fourier's sine series

$$f(x) = \frac{2}{b-a}\sum_{n=1}^{\infty} \sin\left(n\pi\frac{x-a}{b-a}\right)\int_a^b \sin\left(n\pi\frac{y-a}{b-a}\right)f(y)\,dy.$$

Similarly, by taking $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, we obtain the cosine series

$$f(x) = \frac{1}{b-a}\int_a^b f(y)\,dy + \frac{2}{b-a}\sum_{n=1}^{\infty} \cos\left(n\pi\frac{x-a}{b-a}\right)\int_a^b \cos\left(n\pi\frac{y-a}{b-a}\right)f(y)\,dy.$$

1.7. We have now to make the above analysis rigorous. We begin by proving some lemmas.

LEMMA 1.7 (i). *Let $\phi(x) = \phi(x,\lambda)$ be the solution of* (1.1.4) *defined in* § 1.5, *and let $\lambda = s^2$. Then*

$$\phi(x) = \cos\{s(x-a)\}\sin\alpha - \frac{\sin s(x-a)}{s}\cos\alpha + \frac{1}{s}\int_a^x \sin\{s(x-y)\}q(y)\phi(y)\,dy. \qquad (1.7.1)$$

The last term is equal to

$$\frac{1}{s}\int_a^x \sin\{s(x-y)\}\{s^2\phi(y)+\phi''(y)\}\,dy$$

by the differential equation. On integrating by parts twice we obtain

$$\int_a^x \sin\{s(x-y)\}\phi''(y)\,dy = s\phi(x)+\sin\{s(x-a)\}\cos\alpha - s\cos\{s(x-a)\}\sin\alpha - s^2\int_a^x \sin\{s(x-y)\}\phi(y)\,dy.$$

This proves (1.7.1).

LEMMA 1.7 (ii). *For* $s = \sigma+it$, $|s| > s_0$, $\sin\alpha \neq 0$,

$$\phi(x) = O\{e^{|t|(x-a)}\}, \tag{1.7.2}$$

$$\phi(x) = \cos\{s(x-a)\}\sin\alpha+O(|s|^{-1}e^{|t|(x-a)}), \tag{1.7.3}$$

while if $\sin\alpha = 0$,

$$\phi(x) = O(e^{|t|(x-a)}/|s|), \tag{1.7.4}$$

$$\phi(x) = -\frac{\sin s(x-a)}{s}\cos\alpha+O\left(\frac{e^{|t|(x-a)}}{|s|^2}\right). \tag{1.7.5}$$

Each result holds uniformly for $a \leqslant x \leqslant b$.

Putting $\phi(x) = e^{|t|(x-a)}F(x)$, (1.7.1) gives

$$F(x) = \left(\cos\{s(x-a)\}\sin\alpha-\frac{\sin s(x-a)}{s}\cos\alpha\right)e^{-|t|(x-a)}+$$
$$+\frac{1}{s}\int\limits_a^x \sin\{s(x-y)\}e^{-|t|(x-y)}q(y)F(y)\,dy.$$

Let $\mu = \max\limits_{a\leqslant x\leqslant b} |F(x)|$. Then it follows that

$$\mu \leqslant |\sin\alpha|+\frac{|\cos\alpha|}{|s|}+\frac{1}{|s|}\int\limits_a^x |q(y)|\mu\,dy,$$

i.e.

$$\mu \leqslant \frac{|\sin\alpha|+|\cos\alpha|/|s|}{1-\dfrac{1}{|s|}\displaystyle\int\limits_a^b |q(y)|\,dy},$$

provided that the denominator is positive. This is true if $|s|$ is large enough, and (1.7.2) and (1.7.4) follow. Also (1.7.3) follows on substituting (1.7.2) in the integral on the right of (1.7.1); and similarly for (1.7.5).

Of course a general asymptotic expansion of $\phi(x)$ as a function of s can be obtained by repeating the process.

LEMMA 1.7 (iii). *If* $\sin\alpha \neq 0$,

$$\phi'(x) = -s\sin\{s(x-a)\}\sin\alpha+O\{e^{|t|(x-a)}\}, \tag{1.7.6}$$

and if $\sin\alpha = 0$,

$$\phi'(x) = -\cos\{s(x-a)\}\cos\alpha+O(e^{|t|(x-a)}/|s|). \tag{1.7.7}$$

This follows at once on differentiating (1.7.1) and using Lemma 1.7 (ii).

Similarly we obtain

$$\chi(x) \sim \cos\{s(b-x)\}\sin\beta, \qquad \chi'(x) \sim s\sin\{s(b-x)\}\sin\beta \tag{1.7.8}$$

(or similar formulae if $\sin\beta = 0$); and

$$\omega(\lambda) \sim s[\cos\{s(x-a)\}\sin\{s(b-x)\}+\sin\{s(x-a)\}\cos\{s(b-x)\}]\sin\alpha\sin\beta$$
$$= s\sin\{s(b-a)\}\sin\alpha\sin\beta, \tag{1.7.9}$$

or similar formulae in the other cases. It follows in particular that $\omega(\lambda)$ is not identically zero. Actually ω is an integral function of s of order 1, and so an integral function of λ of order $\frac{1}{2}$.

1.8. Orthogonal property of the expansion. If $L = q(x)-d^2/dx^2$, and F and G are any functions of x with continuous second derivatives,

$$\int_a^b F.LG\,dx - \int_a^b G.LF\,dx$$
$$= -\int_a^b (FG''-GF'')\,dx = W_a(F,G)-W_b(F,G). \tag{1.8.1}$$

If F and G satisfy the same boundary conditions at a and at b, so that $F'/F = G'/G$ at these points, it follows that

$$\int_a^b F.LG\,dx = \int_a^b G.LF\,dx. \tag{1.8.2}$$

Now let $F = \phi(x,\lambda)$, $G = \phi(x,\lambda')$ be solutions of (1.1.4) satisfying (1.6.1). Then $W_a(F,G) = 0$. If λ is a zero of $\omega(\lambda)$, the Wronskian of $\phi(x,\lambda)$ and $\chi(x,\lambda)$ vanishes, so that

$$\chi(x,\lambda) = k\phi(x,\lambda),$$

where k is a constant, which is neither 0 nor ∞, by the boundary conditions. Hence by (1.6.1)

$$\phi(b,\lambda) = \frac{\sin\beta}{k}, \qquad \phi'(b,\lambda) = -\frac{\cos\beta}{k}.$$

Similarly, if λ' is a zero of $\omega(\lambda)$, $\phi(x,\lambda')$ satisfies the same conditions at $x = b$. Hence

$$\int_a^b \phi(x,\lambda)L\phi(x,\lambda')\,dx = \int_a^b \phi(x,\lambda')L\phi(x,\lambda)\,dx.$$

Since $L\phi(x,\lambda) = \lambda\phi(x,\lambda)$, $L\phi(x,\lambda') = \lambda'\phi(x,\lambda')$ it follows that

$$(\lambda-\lambda')\int_a^b \phi(x,\lambda)\phi(x,\lambda')\,dx = 0. \tag{1.8.3}$$

Hence, if $\lambda \neq \lambda'$,
$$\int\limits_a^b \phi(x,\lambda)\phi(x,\lambda')\,dx = 0, \tag{1.8.4}$$
the orthogonal property.

If $\lambda = u+iv$ were a complex zero of $\omega(\lambda)$, then so would
$$\lambda' = \bar{\lambda} = u-iv$$
be. Since $\phi(x,\bar{\lambda})$ is the conjugate of $\phi(x,\lambda)$, (1.8.4) gives
$$\int\limits_a^b |\phi(x,\lambda)|^2\,dx = 0,$$
which is impossible if ϕ is not identically zero. Hence all the zeros of $\omega(\lambda)$ are real.

Again, taking $F = \phi(x,\lambda)$, $G = \phi(x,\lambda')$ in (1.8.1), we obtain
$$(\lambda-\lambda')\int\limits_a^b \phi(x,\lambda)\phi(x,\lambda')\,dx = -\phi(b,\lambda)\phi'(b,\lambda')+\phi'(b,\lambda)\phi(b,\lambda').$$
Also
$$\omega(\lambda) = -\phi(b,\lambda)\cos\beta-\phi'(b,\lambda)\sin\beta.$$
Hence, if $\sin\beta \neq 0$,
$$(\lambda-\lambda')\sin\beta\int\limits_a^b \phi(x,\lambda)\phi(x,\lambda')\,dx = \omega(\lambda')\phi(b,\lambda)-\omega(\lambda)\phi(b,\lambda').$$
If λ_0 is a double zero of $\omega(\lambda)$, then $\omega(\lambda_0 \pm iv) = O(v^2)$ as $v \to 0$. Hence, taking $\lambda = \lambda_0+iv$, $\lambda' = \lambda_0-iv$, the right-hand side is $O(v^2)$. But the left-hand side
$$\sim 2iv\sin\beta\int\limits_a^b |\phi(x,\lambda)|^2\,dx,$$
so that we obtain a contradiction. Hence all the zeros of $\omega(\lambda)$ are simple. If $\sin\beta = 0$,
$$(\lambda-\lambda')\int\limits_a^b \phi(x,\lambda)\phi(x,\lambda')\,dx = \pm\{\omega(\lambda)\phi'(b,\lambda')-\omega(\lambda')\phi'(b,\lambda)\},$$
and the result again follows.

It also follows from (1.7.9) that $\omega(\lambda) \neq 0$ for $s = it$ $(t > t_0)$, i.e. for λ negative and sufficiently large.

1.9. THEOREM 1.9. *Let $f(y)$ be integrable over (a,b). Then if $a < x < b$ the Sturm–Liouville expansion* (1.6.5) *behaves as regards convergence in the same way as an ordinary Fourier series. In particular, it converges to $\tfrac{1}{2}\{f(x+0)+f(x-0)\}$ if $f(x)$ is of bounded variation in the neighbourhood of x.*

Consider the integral

$$\frac{1}{2\pi i}\int \Phi(x,\lambda)\,d\lambda, \tag{1.9.1}$$

where $\Phi(x,\lambda)$ is defined by (1.6.2), taken round a large closed contour in the λ-plane. Let $\lambda = s^2$ and let the upper half of the λ-contour correspond to the quarter-square in the s-plane made up by the lines

$$\sigma = \frac{(n+\frac{1}{2})\pi}{b-a} \quad \left(0 \leqslant t \leqslant \frac{(n+\frac{1}{2})\pi}{b-a}\right),$$

$$t = \frac{(n+\frac{1}{2})\pi}{b-a} \quad \left(0 \leqslant \sigma \leqslant \frac{(n+\frac{1}{2})\pi}{b-a}\right),$$

where $s = \sigma+it$; and let the λ-contour be symmetrical about the real axis. Then (1.9.1) is equal to a finite sum of the Sturm–Liouville series.

Consider the case $\sin\alpha \neq 0$, $\sin\beta \neq 0$. Then by Lemma 1.7 (ii)

$$\phi(y,\lambda) = \cos\{(y-a)\}\sin\alpha+O\{|s|^{-1}e^{t(y-a)}\}$$

and

$$\chi(x,\lambda) = \cos\{(b-x)\}\sin\beta+O\{|s|^{-1}e^{t(b-x)}\}$$

on the above quarter-square; and by Lemmas 1.7 (ii) and (iii)

$$\omega(\lambda) = s\sin\{s(b-a)\}\sin\alpha\sin\beta+O\{e^{t(b-a)}\}.$$

On the quarter-square $|\sin\{s(b-a)\}| > Ae^{t(b-a)}$, and hence

$$\frac{1}{\omega(\lambda)} = \frac{1}{s\sin\{s(b-a)\}\sin\alpha\sin\beta}\left\{1+O\left(\frac{1}{|s|}\right)\right\}.$$

Hence

$$\frac{\chi(x,\lambda)\phi(y,\lambda)}{\omega(\lambda)} = \frac{\cos\{s(b-x)\}\cos\{s(y-a)\}}{s\sin\{s(b-a)\}}+O\left(\frac{e^{t(y-x)}}{|s^2|}\right),$$

and

$$\frac{\chi(x,\lambda)}{\omega(\lambda)}\int_a^x \phi(y,\lambda)f(y)\,dy = \int_a^x \frac{\cos\{s(b-x)\}\cos\{s(y-a)\}}{s\sin\{s(b-a)\}}f(y)\,dy+$$

$$+O\left\{\frac{1}{|s|^2}\int_a^x e^{t(y-x)}|f(y)|\,dy\right\}.$$

Let $0 < \delta < x-a$; then the last term is

$$O\left(\frac{e^{-\delta t}}{|s|^2}\right)+O\left\{\frac{1}{|s|^2}\int_{x-\delta}^x |f(y)|\,dy\right\}.$$

Since $d\lambda/ds = 2s$, this contributes to (1.9.1)

$$\int \left\{e^{-\delta t}+\int\limits_{x-\delta}^{x} |f(y)|\, dy\right\} O\left(\frac{1}{|s|}\right) |ds| = O\left\{\int e^{-\delta t}\left|\frac{ds}{s}\right|\right\}+O\left\{\int\limits_{x-\delta}^{x} |f(y)|\, dy\right\}.$$

The second term can be made arbitrarily small by choice of δ; and, having fixed δ, the first term tends to zero as $n \to \infty$; for it is

$$O\left\{\frac{1}{n}\int\limits_{0}^{(n+\frac{1}{2})\pi/(b-a)} e^{-\delta t}\, dt\right\}+O\left\{\frac{1}{n}\int\limits_{0}^{(n+\frac{1}{2})\pi/(b-a)} e^{-\delta(n+\frac{1}{2})\pi/(b-a)}\, d\sigma\right\}$$
$$= O\left(\frac{1}{n\delta}\right)+O(e^{-\delta(n+\frac{1}{2})\pi/(b-a)}).$$

A similar argument applies to the other term in $\Phi(x,\lambda)$ in which $x \leqslant y \leqslant b$. Altogether (1.9.1) is equal to

$$\frac{1}{2\pi i}\int \left\{\int\limits_{a}^{x} \frac{\cos\{s(b-x)\}\cos\{s(y-a)\}}{s\sin\{s(b-a)\}} f(y)\, dy + \right.$$
$$\left. + \int\limits_{x}^{b} \frac{\cos\{s(b-y)\}\cos\{s(x-a)\}}{s\sin\{s(b-a)\}} f(y)\, dy\right\} d\lambda + o(1). \quad (1.9.2)$$

The first term is precisely what we should obtain in the corresponding problem relating to Fourier's cosine series, and consequently it is equal to a finite sum of the Fourier cosine series of $f(x)$. This, of course, is easily verified directly by the calculus of residues. The theorem on the relation between the Sturm–Liouville expansion and the Fourier series therefore follows. If $\sin\alpha = 0$ or $\sin\beta = 0$, the theorem can be proved in a similar way.

In the case in which f is of bounded variation in the neighbourhood of the point x, it is also easy to obtain the result directly, without appeal to the theory of Fourier series. In the first place

$$\frac{\cos\{s(b-x)\}\cos\{s(y-a)\}}{\sin\{s(b-a)\}} = O\left\{\frac{e^{t(b-x)+t(y-a)}}{e^{t(b-a)}}\right\} = O(e^{-t(x-y)}),$$

whence the part of (1.9.2) with $a \leqslant y \leqslant x-\delta$ tends to zero, as before. For $x-\delta \leqslant y \leqslant x$,

$$\frac{\cos\{s(b-x)\}\cos\{s(y-a)\}}{\sin\{s(b-a)\}} = \frac{e^{-is(b-x)}\{1+O(e^{-2t(b-x)})\}e^{-is(y-a)}\{1+O(e^{-2t(y-a)})\}}{2ie^{-is(b-a)}\{1+O(e^{-2t(b-a)})\}}$$
$$= -\tfrac{1}{2}ie^{is(x-y)}\{1+O(e^{-2\delta t})\},$$

if $\delta < b-x$. The contribution of the O-term tends to zero, as before. The main term contributes to the y-integral in (1.9.2):

$$\frac{-i}{2s}\int\limits_{x-\delta}^{x} e^{is(x-y)}f(y)\,dy$$

$$=\frac{-i}{2s}f(x-0)\int\limits_{x-\delta}^{x} e^{is(x-y)}\,dy+\frac{i}{2s}\int\limits_{x-\delta}^{x} e^{is(x-y)}\{f(x-0)-f(y)\}\,dy$$

$$=f(x-0)\frac{1-e^{is\delta}}{2s^2}+\frac{i}{2s}\int\limits_{x-\delta} e^{is(x-y)}\{f(x-0)-f(y)\}\,dy.$$

The first term is $f(x-0)/(2\lambda)$, which contributes $\frac{1}{2}f(x-0)$, as required. The second term is $O(e^{-t\delta}/|\lambda|)$, which gives a zero limit. Also, since f is of bounded variation, we can write $f(x-0)-f(y) = g(y)-h(y)$, where $g(y)$ and $h(y)$ are positive and steadily decreasing, and tend to zero as $y \to x$. By the second mean-value theorem

$$\int\limits_{x-\delta}^{x} \mathbf{R}(e^{is(x-y)})g(y)\,dy = g(x-\delta)\int\limits_{x-\delta}^{\xi} \mathbf{R}e^{is(x-y)}\,dy$$

$$= g(x-\delta)\mathbf{R}\int\limits_{x-\delta}^{\xi} e^{is(x-y)}\,dy = O\left(\frac{g(x-\delta)}{|s|}\right).$$

This contributes to (1.9.1)

$$\int O\left|\frac{g(x-\delta)}{\lambda}\right||d\lambda| = O\{g(x-\delta)\},$$

which can be made as small as we please by choice of δ. Similarly for the term involving $h(y)$, and for the terms with $y > x$, and this proves the theorem.

Writing
$$\psi_n(x) = \left(\frac{k_n}{\omega'(\lambda_n)}\right)^{\frac{1}{2}}\phi(x,\lambda_n),$$
the formulae (1.1.6)–(1.1.9) are now valid.

If $x = a$, similar results hold, and can be proved in the same way, provided that $\sin\alpha \neq 0$. If $\sin\alpha = 0$, $\Phi(a,\lambda)$ vanishes identically, and the result fails unless $f(a+0) = 0$. The situation at $x = b$ is similar.

We also have the following result:

If $f(x)$ is any function integrable over (a,b), and $a \leqslant x \leqslant b$

$$\Phi(x,\lambda) = \sum_{n=0} \frac{c_n\psi_n(x)}{\lambda-\lambda_n} \tag{1.9.3}$$

This easily follows on integrating

$$\int \frac{\Phi(x,z)}{z-\lambda}\,dz$$

round the same contour as before. The series is the Sturm–Liouville expansion of $\Phi(x,\lambda)$, and it is easy to verify that

$$\frac{c_n}{\lambda-\lambda_n} = \int\limits_a^b \Phi(x,\lambda)\psi_n(x)\,dx. \tag{1.9.4}$$

1.10. Example. Let $q(x) = 0$, $\alpha = 0$, $a = 0$, $b > 0$. Then

$$\phi(x,\lambda) = -\frac{\sin x\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \chi(x,\lambda) = \frac{\sin\{(b-x)\sqrt{\lambda}\}}{\sqrt{\lambda}}\cos\beta+\cos\{(b-x)\sqrt{\lambda}\}\sin\beta,$$

and
$$\omega(\lambda) = \frac{1}{\sqrt{\lambda}}\sin(b\sqrt{\lambda})\cos\beta+\cos(b\sqrt{\lambda})\sin\beta.$$

The λ_n are the roots of

$$\frac{\sin(b\sqrt{\lambda})}{\sqrt{\lambda}} = -\tan\beta\cos(b\sqrt{\lambda}).$$

Now
$$\frac{\tan bs}{bs} = -\frac{\tan\beta}{b}$$

has all its roots real if $-\tan\beta < 0$ or $-\tan\beta > b$, and two equal and opposite purely imaginary roots if $0 < -\tan\beta < b$. Also

$$\begin{aligned}\omega'(\lambda_n) &= \frac{b}{2\lambda_n}\cos(b\sqrt{\lambda_n})\cos\beta-\frac{b}{2\sqrt{\lambda_n}}\sin(b\sqrt{\lambda_n})\sin\beta-\frac{1}{2\lambda_n^{\frac{3}{2}}}\sin(b\sqrt{\lambda_n})\cos\beta\\ &= \frac{b\cos\beta\cos(b\sqrt{\lambda_n})}{2\lambda_n}\left\{1-\tan\beta\tan(b\sqrt{\lambda_n})\sqrt{\lambda_n}-\frac{\tan(b\sqrt{\lambda_n})}{b\sqrt{\lambda_n}}\right\}\\ &= \frac{b\cos\beta\cos(b\sqrt{\lambda_n})}{2\lambda_n}\left\{1+\lambda_n\tan^2\beta+\frac{\tan\beta}{b}\right\},\end{aligned}$$

and
$$\begin{aligned}k_n &= \cos\beta\cos(b\sqrt{\lambda_n})-\sqrt{\lambda_n}\sin\beta\sin(b\sqrt{\lambda_n})\\ &= \cos\beta\cos(b\sqrt{\lambda_n})\{1+\lambda_n\tan^2\beta\}.\end{aligned}$$

Hence the expansion is

$$f(x) = \frac{2}{b}\sum_{n=1}^{\infty}\frac{1+\lambda_n\tan^2\beta}{1+\lambda_n\tan^2\beta+\tan\beta/b}\sin(x\sqrt{\lambda_n})\int\limits_0^b \sin(y\sqrt{\lambda_n})f(y)\,dy,$$

where $\sqrt{\lambda_1}$ may be purely imaginary.

In the case $\tan\beta = -b$, we have

$$\omega(\lambda) = \sin\beta\left(\cos b\sqrt{\lambda} - \frac{\sin b\sqrt{\lambda}}{b\sqrt{\lambda}}\right) = -\tfrac{1}{3}b^2\lambda\sin\beta + \dots$$

so that there is a zero at the origin. In the neighbourhood of $\lambda = 0$

$$\phi(x,\lambda) = -x + \dots,$$

$$\chi(x,\lambda) = (b-x)\cos\beta + \sin\beta + \dots = -x\cos\beta + \dots.$$

Hence

$$\Phi(x,\lambda) = \frac{x\cos\beta\int\limits_0^b yf(y)\,dy}{-\frac{1}{3}\sin\beta\,.\,b^2\lambda} + \dots = \frac{3x\int\limits_0^b yf(y)\,dy}{b^3\lambda} + \dots.$$

Hence the first normalized eigenfunction is $\psi_0(x) = 3^{\frac{1}{2}}b^{-\frac{3}{2}}x$.

1.11. An expansion involving Bessel functions. Bessel's equation of order ν is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(s^2 - \frac{\nu^2}{x^2}\right)y = 0, \tag{1.11.1}$$

and the standard solutions are $J_\nu(xs)$, $Y_\nu(xs)$. Putting $y = x^{-\frac{1}{2}}y_1$, we obtain the equation

$$\frac{d^2y_1}{dx^2} + \left(s^2 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)y_1 = 0. \tag{1.11.2}$$

Solutions of this are therefore $x^{\frac{1}{2}}J_\nu(xs)$, $x^{\frac{1}{2}}Y_\nu(xs)$. This equation is of the form considered, with

$$q(x) = \frac{\nu^2 - \frac{1}{4}}{x^2}. \tag{1.11.3}$$

If we consider an interval $a \leqslant x \leqslant b$, where $a > 0$, the conditions of the above theorem are satisfied. Let

$$\phi_0(x,\lambda) = x^{\frac{1}{2}}J_\nu(xs), \qquad \chi_0(x,\lambda) = x^{\frac{1}{2}}Y_\nu(xs),$$

where $s = \sqrt{\lambda}$. Then

$$\omega_0(\lambda) = sx\{J_\nu(xs)Y_\nu'(xs) - Y_\nu(xs)J_\nu'(xs)\} = \frac{2}{\pi}. \tag{1.11.4}$$

Taking $\alpha = 0$ and $\beta = 0$, we have

$$\phi(x,\lambda) = \tfrac{1}{2}\pi(ax)^{\frac{1}{2}}\{J_\nu(xs)Y_\nu(as) - Y_\nu(xs)J_\nu(as)\},$$

$$\chi(x,\lambda) = \tfrac{1}{2}\pi(bx)^{\frac{1}{2}}\{J_\nu(xs)Y_\nu(bs) - Y_\nu(xs)J_\nu(bs)\},$$

and

$$\omega(\lambda) = \tfrac{1}{2}\pi a^{\frac{1}{2}}b^{\frac{1}{2}}\{J_\nu(as)Y_\nu(bs) - Y_\nu(as)J_\nu(bs)\}.$$

Hence

$$\omega'(\lambda) = -\tfrac{1}{2}\pi a^{\frac{1}{2}}b^{\frac{1}{2}}\frac{a}{2s}\{J_\nu(bs)Y_\nu'(as)-J_\nu'(as)Y_\nu(bs)\}-$$

$$-\tfrac{1}{2}\pi a^{\frac{1}{2}}b^{\frac{1}{2}}\frac{b}{2s}\{J_\nu'(bs)Y_\nu(as)-J_\nu(as)Y_\nu'(bs)\}.$$

Substituting for the Y_ν' by means of (1.11.4), we obtain

$$\omega'(\lambda) = -\tfrac{1}{2}\pi a^{\frac{1}{2}}b^{\frac{1}{2}}\frac{a}{2s}\left[J_\nu(bs)\left\{\frac{Y_\nu(as)J_\nu'(as)}{J_\nu(as)}+\frac{2}{\pi asJ_\nu(as)}\right\}-J_\nu'(as)Y_\nu(bs)\right]-$$

$$-\tfrac{1}{2}\pi a^{\frac{1}{2}}b^{\frac{1}{2}}\frac{b}{2s}\left[J_\nu'(bs)Y_\nu(as)-J_\nu(as)\left\{\frac{Y_\nu(bs)J_\nu'(bs)}{J_\nu(bs)}+\frac{2}{\pi bsJ_\nu(bs)}\right\}\right]$$

$$= -\left\{\frac{a}{2s}\frac{J_\nu'(as)}{J_\nu(as)}+\frac{b}{2s}\frac{J_\nu'(bs)}{J_\nu(bs)}\right\}\omega(\lambda)-\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{2s^2}\left\{\frac{J_\nu(bs)}{J_\nu(as)}-\frac{J_\nu(as)}{J_\nu(bs)}\right\}.$$

Hence if the zeros of $\omega(\lambda)$ are λ_n, and $\sqrt{\lambda_n} = s_n$,

$$\omega'(\lambda_n) = -\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{2s_n^2}\frac{J_\nu^2(bs_n)-J_\nu^2(as_n)}{J_\nu(as_n)J_\nu(bs_n)}.$$

Also
$$k_n = \left(\frac{b}{a}\right)^{\frac{1}{2}}J_\nu(bs_n)\Big/J_\nu(as_n).$$

The expansion obtained is therefore

$$f(x) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 J_\nu^2(bs_n)}{J_\nu^2(as_n)-J_\nu^2(bs_n)}x^{\frac{1}{2}}\{J_\nu(xs_n)Y_\nu(as_n)-Y_\nu(xs_n)J_\nu(as_n)\}\times$$

$$\times\int_a^b y^{\frac{1}{2}}\{J_\nu(ys_n)Y_\nu(as_n)-Y_\nu(ys_n)J_\nu(as_n)\}f(y)\,dy.$$

A similar result can be obtained from the more general boundary conditions.

REFERENCES

Birkhoff (1), (2), Dixon (1), (2), Haar (1), (2), Hilb (1), (2), Hilbert (1), Hobson (1), Kneser (1), (2), (3), Lichtenstein (1), Prüfer (1), Schur (1), Tamarkin (1), (2), Titchmarsh (2), Zaanen (1), Zygmund (1).

II

THE SINGULAR CASE: SERIES EXPANSIONS

2.1. In many of the most interesting examples the function $q(x)$ has a singularity at one end or both ends of the interval considered, or the interval extends to infinity in one direction or in both. We shall now consider these cases, but, in this chapter, with a limitation which makes the expansion still a series. The result is a particular case of the more general one obtained in the next chapter, but it seems worth while to prove it separately on account of its comparative simplicity.

We begin by considering the case in which the interval is $(0,\infty)$, $q(x)$ being continuous over any finite interval. The case of a finite interval with a singularity at one end is quite similar to this.

If $F(x)$ and $G(x)$ satisfy the differential equations

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0, \tag{2.1.1}$$

and the corresponding equation with λ' instead of λ, respectively, then

$$\begin{aligned}(\lambda'-\lambda)\int\limits_0^b F(x)G(x)\,dx &= \int\limits_0^b [F(x)\{q(x)G(x)-G''(x)\}-\\ &\qquad\qquad -G(x)\{q(x)F(x)-F''(x)\}]\,dx\\ &= -\int\limits_0^b \{F(x)G''(x)-G(x)F''(x)\}\,dx\\ &= W_0(F,G)-W_b(F,G).\end{aligned} \tag{2.1.2}$$

If $\lambda = u+iv$, $\lambda' = \bar\lambda = u-iv$, and $G = \bar F$, this gives

$$2v\int\limits_0^b |F(x)|^2\,dx = iW_0(F,\bar F)-iW_b(F,\bar F). \tag{2.1.3}$$

Now let $\phi(x) = \phi(x,\lambda)$, $\theta(x) = \theta(x,\lambda)$ be the solutions of (2.1.1) such that

$$\left.\begin{aligned}\phi(0) &= \sin\alpha, & \phi'(0) &= -\cos\alpha,\\ \theta(0) &= \cos\alpha, & \theta'(0) &= \sin\alpha,\end{aligned}\right\} \tag{2.1.4}$$

where α is real. Then

$$W_x(\phi,\theta) = W_0(\phi,\theta) = \sin^2\alpha+\cos^2\alpha = 1.$$

The general solution of (2.1.1) is of the form $\theta(x)+l\phi(x)$. Consider those solutions which satisfy a real boundary condition at $x = b$, say

$$\{\theta(b)+l\phi(b)\}\cos\beta+\{\theta'(b)+l\phi'(b)\}\sin\beta = 0,$$

where β is real. This gives

$$l = l(\lambda) = -\frac{\theta(b)\cot\beta+\theta'(b)}{\phi(b)\cot\beta+\phi'(b)}. \tag{2.1.5}$$

For each b, as $\cot\beta$ varies, l describes a circle in the complex plane, say C_b. Replacing $\cot\beta$ by a complex variable z,

$$l = l(\lambda, z) = -\frac{\theta(b)z+\theta'(b)}{\phi(b)z+\phi'(b)}.$$

Here $l = \infty$ corresponds to $z = -\phi'(b)/\phi(b)$. Hence the centre of C_b corresponds to the conjugate, $z = -\bar{\phi}'(b)/\bar{\phi}(b)$; it is therefore

$$-\frac{W_b(\theta, \bar{\phi})}{W_b(\phi, \bar{\phi})}.$$

Also $$\mathbf{I}\left\{-\frac{\phi'(b)}{\phi(b)}\right\} = \tfrac{1}{2}i\left\{\frac{\phi'(b)}{\phi(b)}-\frac{\bar{\phi}'(b)}{\bar{\phi}(b)}\right\} = -\tfrac{1}{2}i\frac{W_b(\phi,\bar{\phi})}{|\phi(b)|^2},$$

which has the same sign as v, by (2.1.3) with $F = \phi$, since

$$W_0(\phi,\bar{\phi}) = 0.$$

Hence, if $v > 0$, the exterior of C_b corresponds to the upper half of the z-plane.

Since $-\theta'(b)/\phi'(b)$ is on C_b (for $z = 0$) the radius r_b of C_b is

$$r_b = \left|\frac{\theta'(b)}{\phi'(b)}-\frac{W_b(\theta,\bar{\phi})}{W_b(\phi,\bar{\phi})}\right| = \left|\frac{W_b(\theta,\phi)}{W_b(\phi,\bar{\phi})}\right| = \frac{1}{2v\int\limits_0^b |\phi(x)|^2\,dx}. \tag{2.1.6}$$

Now l is inside C_b if $\mathbf{I}(z) < 0$, i.e. if $i(z-\bar{z}) > 0$, i.e. if

$$i\left\{-\frac{l\phi'(b)+\theta'(b)}{l\phi(b)+\theta(b)}+\frac{\bar{l}\bar{\phi}'(b)+\bar{\theta}'(b)}{\bar{l}\bar{\phi}(b)+\bar{\theta}(b)}\right\} > 0,$$

i.e. if $$i\{|l|^2W_b(\phi,\bar{\phi})+lW_b(\phi,\bar{\theta})+\bar{l}W_b(\theta,\bar{\phi})+W_b(\theta,\bar{\theta})\} > 0,$$

i.e. if $$iW_b(\theta+l\phi,\bar{\theta}+\bar{l}\bar{\phi}) > 0,$$

i.e. (by (2.1.3)) if

$$2v\int\limits_0^b |\theta+l\phi|^2\,dx < iW_0(\theta+l\phi,\bar{\theta}+\bar{l}\bar{\phi}).$$

Since $W_0(\phi,\theta) = 1$, $W_0(\phi,\bar{\phi}) = 0$, etc.,

$$W_0(\theta+l\phi,\bar{\theta}+\bar{l}\bar{\phi}) = l-\bar{l} = 2i\,\mathbf{I}(l).$$

Hence l is interior to C_b if $v > 0$, and

$$\int\limits_0^b |\theta+l\phi|^2\,dx < -\frac{\mathbf{I}(l)}{v}. \tag{2.1.7}$$

The same result is obtained if $v < 0$. In each case the sign of $\mathbf{I}(l)$ is opposite to that of v.

It follows that, if l is inside C_b, and $0 < b' < b$, then

$$\int_0^{b'} |\theta+l\phi|^2\,dx < \int_0^b |\theta+l\phi|^2\,dx < -\frac{\mathbf{I}(l)}{v}.$$

Hence l is also inside $C_{b'}$. Hence $C_{b'}$ includes C_b if $b' < b$. It follows that, as $b \to \infty$, the circles C_b converge either to a limit-circle or to a limit-point.

If $m = m(\lambda)$ is the limit-point, or any point on the limit-circle,

$$\int_0^b |\theta+m\phi|^2\,dx \leqslant -\frac{\mathbf{I}(m)}{v} \tag{2.1.8}$$

for all values of b. Hence

$$\int_0^\infty |\theta+m\phi|^2\,dx \leqslant -\frac{\mathbf{I}(m)}{v}. \tag{2.1.9}$$

It follows that, *for every value of λ other than real values,* (2.1.1) *has a solution*

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

belonging to $L^2(0,\infty)$.

In the limit-circle case, r_b tends to a positive limit as $b \to \infty$; hence, by (2.1.6), $\phi(x)$ is $L^2(0,\infty)$; so in fact, in this case, every solution of (2.1.1) belongs to $L^2(0,\infty)$.

2.2. For a given β, $l = l(\lambda)$ is an analytic function of λ; in fact it is a meromorphic function, regular except for poles on the real axis. For the poles of $l(\lambda)$ are the zeros of

$$\phi(b,\lambda)\cos\beta+\phi'(b,\lambda)\sin\beta,$$

and this is $-\omega(\lambda)$ in the notation of § 1.6.

Also on the circle C_b (if $v > 0$)

$$\int_0^b |\theta+l\phi|^2\,dx = -\frac{\mathbf{I}(l)}{v} \leqslant \frac{|l|}{v}$$

and

$$\geqslant |l|^2 \int_0^b |\phi|^2\,dx - \int_0^b |\theta|^2\,dx.$$

Solving for $|l|$, this gives

$$|l| \leqslant \frac{1}{2v\int_0^b |\phi|^2\,dx} + \left\{\frac{\int_0^b |\theta|^2\,dx}{\int_0^b |\phi|^2\,dx} + \frac{1}{4v^2\left(\int_0^b |\phi|^2\,dx\right)^2}\right\}^{\frac{1}{2}}$$

But the above argument shows that, for a given λ, the region of the l-plane covered by the circles C_b decreases as b increases. Hence $l(\lambda)$ converges boundedly in any region entirely in the upper (or lower) half of the λ-plane. Hence its limit $m(\lambda)$ is analytic in either half-plane.

Since the above right-hand side is $O(1/v)$ as $v \to 0$, for any fixed b, it also follows that $m(\lambda) = O(1/v)$. Hence, if $m(\lambda)$ has poles on the real axis, they are all simple.

We shall assume in this chapter that *the only singularities of* $m(\lambda)$ *are poles*. Let them be $\lambda_0, \lambda_1, \lambda_2,\ldots$, and let the residues be $r_0, r_1, r_2,\ldots$.

2.3. LEMMA 2.3. *For any fixed complex* λ *and* λ'

$$\lim_{x\to\infty} W\{\psi(x,\lambda), \psi(x,\lambda')\} = 0. \tag{2.3.1}$$

Since $\theta(x,\lambda)+l(\lambda)\phi(x,\lambda)$ satisfies at $x = b$ a boundary condition independent of λ,

$$W_b\{\theta(x,\lambda)+l(\lambda)\phi(x,\lambda), \theta(x,\lambda')+l(\lambda')\phi(x,\lambda')\} = 0,$$

i.e.

$$W_b[\psi(x,\lambda)+\{l(\lambda)-m(\lambda)\}\phi(x,\lambda), \psi(x,\lambda')+\{l(\lambda')-m(\lambda')\}\phi(x,\lambda')] = 0,$$

i.e.

$$\begin{aligned} W_b\{\psi(x,\lambda), \psi(x,\lambda')\}+\{l(\lambda)-m(\lambda)\}W_b\{\phi(x,\lambda), \psi(x,\lambda')\}+ \\ +\{l(\lambda')-m(\lambda')\}W_b\{\psi(x,\lambda), \phi(x,\lambda')\}+ \\ +\{l(\lambda)-m(\lambda)\}\{l(\lambda')-m(\lambda')\}W_b\{\phi(x,\lambda), \phi(x,\lambda')\} = 0. \end{aligned} \tag{2.3.2}$$

Now

$$\begin{aligned} W_b\{\phi(x,\lambda), \psi(x,\lambda')\} &= (\lambda'-\lambda) \int_0^b \phi(x,\lambda)\psi(x,\lambda')\,dx + W_0\{\phi(x,\lambda)\psi(x,\lambda')\} \\ &= O\left\{\int_0^b |\phi(x,\lambda)|^2\,dx\right\}^{\frac{1}{2}} + O(1) \end{aligned}$$

as $b \to \infty$, λ and λ' being fixed. In the limit-point case

$$|l(\lambda)-m(\lambda)| \leqslant 2r_b = \left\{v \int_0^b |\phi(x,\lambda)|^2\,dx\right\}^{-1},$$

so that

$$\lim_{b\to\infty} \{l(\lambda)-m(\lambda)\}W_b\{\phi(x,\lambda), \psi(x,\lambda')\} = 0.$$

This also holds in the limit-circle case, if $l(\lambda) \to m(\lambda)$, since then $\int_0^b |\phi(x,\lambda)|^2\,dx$ is bounded. Similar arguments apply to the other terms in (2.3.2), and (2.3.1) follows.

2.4. The following general lemma on sequences of integrals will be used.

LEMMA 2.4. *Let $f_\nu(x)$ be a sequence of functions which converges in mean square to $f(x)$ over any finite interval, while*

$$\int_0^\infty |f_\nu(x)|^2\,dx \leqslant K$$

for all ν. Then $f(x)$ is $L^2(0,\infty)$, and if $g(x)$ belongs to $L^2(0,\infty)$,

$$\lim_{\nu\to\infty}\int_0^\infty f_\nu(x)g(x)\,dx = \int_0^\infty f(x)g(x)\,dx.$$

We have

$$\int_0^X |f(x)|^2\,dx = \lim_{\nu\to\infty}\int_0^X |f_\nu(x)|^2\,dx \leqslant K$$

for every X, so that $f(x)$ is $L^2(0,\infty)$. Now

$$\left|\int_0^\infty (f-f_\nu)g\,dx\right| \leqslant \left|\int_0^X\right|+\left|\int_X^\infty\right| \leqslant \left\{\int_0^X |f-f_\nu|^2\,dx\int_0^\infty |g|^2\,dx\right\}^{\frac{1}{2}}+$$

$$+\left\{\int_0^\infty |f-f_\nu|^2\,dx\int_X^\infty |g|^2\,dx\right\}^{\frac{1}{2}}.$$

The second term can be made less than any given ϵ, for all ν, by choice of X. Having fixed X, the first term tends to zero as $\nu\to\infty$, and the result follows.

2.5. By (2.1.2)

$$(\lambda'-\lambda)\int_0^b \psi(x,\lambda)\psi(x,\lambda')\,dx = W_0\{\psi(x,\lambda),\psi(x,\lambda')\}-W_b\{\psi(x,\lambda),\psi(x,\lambda')\}.$$

The first term on the right is

$$\{\cos\alpha+m(\lambda)\sin\alpha\}\{\sin\alpha-m(\lambda')\cos\alpha\}-$$
$$-\{\cos\alpha+m(\lambda')\sin\alpha\}\{\sin\alpha-m(\lambda)\cos\alpha\} = m(\lambda)-m(\lambda'),$$

and, if λ and λ' are not real, the second term tends to zero as $b\to\infty$, by Lemma 2.3. Hence

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda')\,dx = \frac{m(\lambda)-m(\lambda')}{\lambda'-\lambda}. \tag{2.5.1}$$

In particular, taking $\lambda' = \bar{\lambda}$,

$$\int_0^\infty |\psi(x,\lambda)|^2\, dx = -\frac{\mathrm{I}\{m(\lambda)\}}{v}, \tag{2.5.2}$$

so that the case of equality holds in (2.1.9).

Now let λ_n be an eigenvalue, and let $\lambda' = \lambda_n + iv$, $v \to 0$. Then for any fixed X

$$\int_0^X |v\psi(x,\lambda') + ir_n\,\phi(x,\lambda_n)|^2\, dx \to 0.$$

For the left-hand side is

$$\int_0^X |v\theta(x,\lambda') + \{vm(\lambda') + ir_n\}\phi(x,\lambda') - ir_n\{\phi(x,\lambda') - \phi(x,\lambda_n)\}|^2\, dx$$

and each of the three terms obviously contributes zero to the limit. Also, by (2.5.2),

$$\int_0^\infty |v\psi(x,\lambda')|^2\, dx \leqslant |vm(\lambda')| = O(1)$$

as $v \to 0$, since the pole of $m(\lambda')$ at λ_n is simple. On multiplying (2.5.1) by iv/r_n, making $v \to 0$, and using Lemma 2.4, it follows that $\phi(x,\lambda_n)$ is $L^2(0,\infty)$, and

$$\int_0^\infty \psi(x,\lambda)\phi(x,\lambda_n)\, dx = \frac{1}{\lambda - \lambda_n}. \tag{2.5.3}$$

If λ tends to a different eigenvalue λ_m, on multiplying (2.5.3) by iv/r_m and making $v \to 0$, we obtain

$$\int_0^\infty \phi(x,\lambda_m)\phi(x,\lambda_n)\, dx = 0. \tag{2.5.4}$$

If λ tends to the same eigenvalue λ_n, it follows similarly that

$$\int_0^\infty \{\phi(x,\lambda_n)\}^2\, dx = \frac{1}{r_n}. \tag{2.5.5}$$

Hence the functions

$$\psi_n(x) = r_n^{\frac{1}{2}}\phi(x,\lambda_n) \tag{2.5.6}$$

form a normal orthogonal set.

(2.5.3) can now be written as

$$\int_0^\infty \psi(x,\lambda)\psi_n(x)\, dx = \frac{r_n^{\frac{1}{2}}}{\lambda - \lambda_n}. \tag{2.5.7}$$

2.6. The function $\Phi(x,\lambda)$. Let $f(y)$ be $L^2(0,\infty)$, and

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy, \tag{2.6.1}$$

where ϕ and ψ are the functions defined above. Then, for every x, $\Phi(x,\lambda)$ is an analytic function of λ, regular for $\mathbf{I}(\lambda)>0$ or $\mathbf{I}(\lambda)<0$. Also, if $f(y)$ is continuous,

$$\Phi'(x,\lambda) = \psi'(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy+\phi'(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy$$

and

$$\begin{aligned}\Phi''(x,\lambda) &= \psi''(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy+\phi''(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy+\\ &\qquad+\{\psi'(x,\lambda)\phi(x,\lambda)-\phi'(x,\lambda)\psi(x,\lambda)\}f(x)\\ &= \{q(x)-\lambda\}\Phi(x,\lambda)+f(x).\end{aligned} \tag{2.6.2}$$

Thus $\Phi(x,\lambda)$ satisfies the differential equation suggested by §§ 1.3–1.4; and

$$\Phi(0,\lambda) = \phi(0,\lambda)\int_0^\infty \psi(y,\lambda)f(y)\,dy,$$

$$\Phi'(0,\lambda) = \phi'(0,\lambda)\int_0^\infty \psi(y,\lambda)f(y)\,dy,$$

so that Φ satisfies the boundary condition

$$\Phi(0,\lambda)\cos\alpha+\Phi'(0,\lambda)\sin\alpha = 0. \tag{2.6.3}$$

If $\Phi_X(x,\lambda)$ is the corresponding function with $f(y)=0$ for $y>X$, then

$$\begin{aligned}\Phi_X(x,\lambda) &= \theta(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int_x^X \theta(y,\lambda)f(y)\,dy+\\ &\qquad+m(\lambda)\phi(x,\lambda)\int_0^X \phi(y,\lambda)f(y)\,dy.\end{aligned}$$

This is clearly regular everywhere except at $\lambda=\lambda_0,\lambda_1,\ldots$, where it has simple poles with residues

$$r_n\phi(x,\lambda_n)\int_0^X \phi(y,\lambda_n)f(y)\,dy.$$

Let C denote a rectangle of the form

$$\xi_1 \leqslant \mathbf{R}(z) \leqslant \xi_2, \qquad -\eta \leqslant \mathbf{I}(z) \leqslant \eta,$$

excluding all λ_n's. If λ is in C,

$$\Phi_X(x,\lambda) = \frac{1}{2\pi i}\int\limits_C \frac{\Phi_X(x,z)}{z-\lambda}\,dz.$$

Now $\Phi_X(x,\lambda) \to \Phi(x,\lambda)$ if λ is not real; and as $v = \mathbf{I}(\lambda) \to 0$

$$\Phi_X(x,\lambda) = O\left\{\int\limits_x^\infty \psi(y,\lambda)f(y)\,dy\right\} = O\left\{\int\limits_0^\infty |\psi(y,\lambda)|^2\,dy\right\}^{\frac{1}{2}} = O(v^{-\frac{1}{2}})$$

by (2.5.2), if λ tends to a value not an eigenvalue. Hence making $X \to \infty$ we obtain by dominated convergence

$$\Phi(x,\lambda) = \frac{1}{2\pi i}\int\limits_C \frac{\Phi(x,z)}{z-\lambda}\,dz.$$

It follows that $\Phi(x,\lambda)$ is analytic throughout C, i.e. that the functions so denoted in the upper and lower half-planes are analytic continuations of each other. If C includes a point λ_n, we find similarly that $\Phi(x,\lambda)$ has a simple pole at λ_n, its residue there being the limit of the residue of $\Phi_X(x,\lambda)$, i.e.

$$r_n\,\phi(x,\lambda_n)\int\limits_0^\infty \phi(y,\lambda_n)f(y)\,dy = \psi_n(x)\int\limits_0^\infty \psi_n(y)f(y)\,dy = c_n\,\psi_n(x).$$

2.7. The following two theorems will now be proved.

THEOREM 2.7 (i). *Let $f(x)$ and*

$$L\{f(x)\} = q(x)f(x)-f''(x)$$

be $L^2(0,\infty)$; let

$$f(0)\cos\alpha+f'(0)\sin\alpha = 0 \tag{2.7.1}$$

and

$$\lim_{x\to\infty} W\{\psi(x,\lambda),f(x)\} = 0 \tag{2.7.2}$$

for every non-real λ. Then

$$f(x) = \sum_{n=0}^{\infty} c_n\,\psi_n(x) \quad (0 \leqslant x < \infty), \tag{2.7.3}$$

the series being absolutely and uniformly convergent in any finite interval.

THEOREM 2.7 (ii). *Let $f(x)$ be $L^2(0,\infty)$. Then*

$$\int\limits_0^\infty \{f(x)\}^2\,dx = \sum_{n=0}^{\infty} c_n^2. \tag{2.7.4}$$

Either theorem can be used as a means of proving the other. We shall first prove 2.7 (ii) and deduce 2.7 (i), and then give the alternative procedure.

We require a number of lemmas.

2.8. LEMMA 2.8. *If $f(x)$ is any function of $L^2(0,\infty)$,*

$$\int_0^\infty |\Phi(x,\lambda)|^2\,dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\,dx. \tag{2.8.1}$$

Suppose first that $f(x)=0$ for $x \geqslant X$. Then the 'condition of self-adjointness'

$$\int_0^\infty \Phi(x,\lambda)L\Phi(x,\lambda')\,dx = \int_0^\infty \Phi(x,\lambda')L\Phi(x,\lambda)\,dx \tag{2.8.2}$$

is satisfied. For

$$\int_0^\xi \{\Phi(x,\lambda)L\Phi(x,\lambda')-\Phi(x,\lambda')L\Phi(x,\lambda)\}\,dx$$
$$= -\int_0^\xi \{\Phi(x,\lambda)\Phi''(x,\lambda')-\Phi(x,\lambda')\Phi''(x,\lambda)\}\,dx$$
$$= -[\Phi(x,\lambda)\Phi'(x,\lambda')-\Phi(x,\lambda')\Phi'(x,\lambda)]_0^\xi.$$

The integrated term vanishes at $x=0$, since

$$\Phi(0,\lambda) = \sin\alpha\int_0^\infty \psi(y,\lambda)f(y)\,dy,$$

$$\Phi'(0,\lambda) = -\cos\alpha\int_0^\infty \psi(y,\lambda)f(y)\,dy.$$

The integrated term at $x=\xi$ tends to 0 as $\xi\to\infty$, since, if $x>X$,

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^X \phi(y,\lambda)f(y)\,dy,$$

$$\Phi'(x,\lambda) = \psi'(x,\lambda)\int_0^X \phi(y,\lambda)f(y)\,dy,$$

and the result follows from Lemma 2.3.

Putting $\lambda'=\bar\lambda$ in (2.8.2), and substituting from (2.6.2), we obtain

$$\int_0^\infty \Phi(x,\lambda)\{\bar\lambda\Phi(x,\bar\lambda)-f(x)\}\,dx = \int_0^\infty \Phi(x,\bar\lambda)\{\lambda\Phi(x,\lambda)-f(x)\}\,dx,$$

i.e. $$(\lambda-\bar{\lambda})\int_0^\infty |\Phi(x,\lambda)|^2\,dx = \int_0^\infty \{\Phi(x,\bar{\lambda})-\Phi(x,\lambda)\}f(x)\,dx.$$

Hence, if $\lambda = u+iv$, $v>0$,

$$2v\int_0^\infty |\Phi(x,\lambda)|^2\,dx \leqslant 2\int_0^\infty |\Phi(x,\lambda)f(x)|\,dx$$

$$\leqslant 2\left\{\int_0^\infty |\Phi(x,\lambda)|^2\,dx \int_0^\infty |f(x)|^2\,dx\right\}^{\frac{1}{2}}.$$

Hence $$\int_0^\infty |\Phi(x,\lambda)|^2\,dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\,dx,$$

the required result in the restricted case.

If now $f(x)$ is any function of $L^2(0,\infty)$, and the above functions are distinguished by a suffix X,

$$\lim_{X\to\infty}\int_0^{X'} |\Phi_X(x,\lambda)|^2\,dx = \int_0^{X'} |\Phi(x,\lambda)|^2\,dx$$

by uniform convergence, for a fixed X'. Now

$$\int_0^{X'} |\Phi_X(x,\lambda)|^2\,dx \leqslant \int_0^\infty |\Phi_X(x,\lambda)|^2\,dx$$

$$\leqslant \frac{1}{v^2}\int_0^\infty |f_X(x)|^2\,dx$$

$$= \frac{1}{v^2}\int_0^X |f(x)|^2\,dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\,dx.$$

The result therefore follows on making first $X\to\infty$, then $X'\to\infty$.

2.9. LEMMA 2.9. *If $f(x)$ satisfies the conditions of Theorem 2.7 (i), then*

$$\Phi(x,\lambda) = \frac{1}{\lambda}\{f(x)+\Phi^*(x,\lambda)\}, \tag{2.9.1}$$

where Φ^* depends on Lf in the same way as Φ depends on f.

We have

$$\int_0^x \phi(y,\lambda)f(y)\,dy = \frac{1}{\lambda}\int_0^x \{q(y)\phi(y)-\phi''(y)\}f(y)\,dy$$

$$= \frac{1}{\lambda}[\phi(y)f'(y)-\phi'(y)f(y)]_0^x+\frac{1}{\lambda}\int_0^x \{q(y)f(y)-f''(y)\}\phi(y)\,dy$$

on integrating by parts twice; and the integrated term vanishes at the lower limit, by (2.7.1). Similarly

$$\int_x^\infty \psi(y,\lambda)f(y)\,dy = \frac{1}{\lambda}[\psi(y)f'(y)-\psi'(y)f(y)]_x^\infty +$$

$$+\frac{1}{\lambda}\int_x^\infty \{q(y)f(y)-f''(y)\}\psi(y)\,dy,$$

and the integrated term vanishes at the upper limit, by (2.7.2). Substituting in (2.6.1), the result follows.

If $G(x,y,\lambda)$ denotes the 'Green's function'

$$G(x,y,\lambda) = \psi(x,\lambda)\phi(y,\lambda) \quad (y \leqslant x), \qquad \phi(x,\lambda)\psi(y,\lambda) \quad (y > x),$$

then

$$\Phi(x,\lambda) = \int_0^\infty G(x,y,\lambda)f(y)\,dy,$$

$$\Phi^*(x,\lambda) = \int_0^\infty G(x,y,\lambda)L\{f(y)\}\,dy,$$

and (2.9.1) is

$$f(x) = \int_0^\infty G(x,y,\lambda)[\lambda f(y)-L\{f(y)\}]\,dy. \tag{2.9.2}$$

2.10. LEMMA 2.10. *If $f(x)$ is $L^2(0,\infty)$,*

$$\sum_{n=0}^\infty c_n^2 \leqslant \int_0^\infty \{f(x)\}^2\,dx.$$

This is the 'Bessel's inequality'. We have

$$0 \leqslant \int_0^\infty \Big\{f(x)-\sum_{n=0}^N c_n\psi_n(x)\Big\}^2\,dx$$

$$= \int_0^\infty \{f(x)\}^2\,dx + \sum_{n=0}^N c_n^2 - 2\sum_{n=0}^N c_n\int_0^\infty f(x)\psi_n(x)\,dx$$

$$= \int_0^\infty \{f(x)\}^2\,dx - \sum_{n=0}^N c_n^2$$

for every N, and the result follows.

2.11. LEMMA 2.11. *Let $F(\lambda)$ be an analytic function of $\lambda = u+iv$, regular for $-r \leqslant u \leqslant r$, $-r \leqslant v \leqslant r$, and let*

$$|F(\lambda)| \leqslant \frac{M}{|v|}$$

in this square. Then

$$|F(\lambda)| \leqslant \frac{3M}{r} \quad (u = 0,\ -r \leqslant v \leqslant r).$$

Let
$$G(\lambda) = (\lambda^2-r^2)F(\lambda).$$

On the upper and lower sides of the square

$$|G(\lambda)| \leqslant (|\lambda|^2+r^2)\,\frac{M}{r} \leqslant 3rM.$$

On the left- and right-hand sides

$$|G(\lambda)| \leqslant |v|(|\lambda|+r)\,\frac{M}{|v|} \leqslant 3rM.$$

Hence $|G(\lambda)| \leqslant 3rM$ throughout the square. Hence on the imaginary axis

$$|F(\lambda)| \leqslant \frac{3rM}{|\lambda^2-r^2|} = \frac{3rM}{v^2+r^2} \leqslant \frac{3M}{r}.$$

2.12. Proof of Theorem 2.7 (ii). Suppose first that $f(x)$ satisfies the conditions of Theorem 2.7 (i), and also that $f(x) = 0$ for sufficiently large values of x. Let

$$\Psi(\lambda) = \int\limits_0^\infty f(x)\Phi(x,\lambda)\,dx, \tag{2.12.1}$$

this being really an integral over a finite range. Then $\Psi(\lambda)$ is regular except for simple poles at the points λ_n, where it has residues

$$c_n \int\limits_0^\infty \psi_n(x)f(x)\,dx = c_n^2.$$

By (2.9.1)

$$\Psi(\lambda) = \frac{1}{\lambda}\int\limits_0^\infty \{f(x)\}^2\,dx + \frac{1}{\lambda}\int\limits_0^\infty \Phi^*(x,\lambda)f(x)\,dx, \tag{2.12.2}$$

and the last term is

$$O\left\{\frac{1}{|\lambda|}\left[\int\limits_0^\infty |\Phi^*(x,\lambda)|^2\,dx \int\limits_0^\infty \{f(x)\}^2\,dx\right]^{\frac{1}{2}}\right\} = O\left(\frac{1}{|\lambda v|}\right)$$

by Lemma 2.8, applied to Lf.

Let $C(R)$ denote the contour formed by the segments of lines $(R-i, R+i)$ and $(-R-i, -R+i)$, joined by semicircles of radius R and centres $\pm i$. Then

$$\int\limits_{C(R)} \Psi(\lambda)\, d\lambda = 2\pi i \sum_{-R<\lambda_n<R} c_n^2$$

if none of the λ_n is equal to $\pm R$.

On the part of the upper semicircle in the first quadrant, we have

$$\lambda = i+Re^{i\phi} \quad (0 \leqslant \phi \leqslant \tfrac{1}{2}\pi).$$

Hence the last term in (2.12.2), integrated round this quadrant, gives

$$O\left\{\int\limits_0^{\frac{1}{2}\pi} \frac{R\, d\phi}{R(1+R\sin\phi)}\right\} = O\left\{\int\limits_0^{1/R} d\phi\right\}+O\left\{\int\limits_{1/R}^{\frac{1}{2}\pi} \frac{d\phi}{R\phi}\right\}$$

$$= O\left(\frac{1}{R}\right)+O\left(\frac{\log R}{R}\right) = o(1).$$

A similar argument applies to the other quadrants. Hence the integral of $\Psi(\lambda)$ round each semicircle tends to

$$\pi i \int\limits_0^\infty \{f(x)\}^2\, dx$$

as $R \to \infty$. To prove the theorem for the class of functions considered, it is therefore sufficient to prove that

$$\lim_{R\to\infty} \int\limits_{R-i}^{R+i} \Psi(\lambda)\, d\lambda = 0$$

and a similar result with $-R$ in place of R.

Let $$\chi(\lambda) = \Psi(\lambda)-\sum_{R-1\leqslant\lambda_n\leqslant R+1} \frac{c_n^2}{\lambda-\lambda_n}.$$

Then $\chi(\lambda)$ is regular for $R-1 \leqslant \lambda \leqslant R+1$, and

$$|\chi(\lambda)| < \frac{K}{|\lambda v|}+\frac{1}{|v|}\sum_{R-1\leqslant\lambda_n\leqslant R+1} c_n^2 \leqslant \frac{\epsilon(R)}{|v|},$$

where $\epsilon(R) \to 0$ as $R \to \infty$. Hence, by Lemma 2.11,

$$|\chi(\lambda)| \leqslant 3\epsilon(R)$$

on the segment $(R-i, R+i)$. Hence

$$\lim_{R\to\infty} \int\limits_{R-i}^{R+i} \chi(\lambda)\, d\lambda = 0.$$

Also
$$\int_{R-i}^{R+i} \frac{d\lambda}{\lambda-\lambda_n} = O(1),$$
since the path of integration can be replaced by a semicircle on the side opposite to λ_n, and on this semicircle the integrand is bounded. Hence
$$\int_{R-i}^{R+i} \sum_{R-1\leqslant\lambda_n\leqslant R+1} \frac{c_n^2}{\lambda-\lambda_n}\, d\lambda = O\Big\{\sum_{R-1\leqslant\lambda_n\leqslant R+1} c_n^2\Big\} = o(1).$$
This proves the theorem for the special class of functions.

Now let $f(x)$ be any function of $L^2(0,\infty)$. Then we can determine a function $f_0(x)$ of the special class such that
$$\int_0^\infty \{f(x)-f_0(x)\}^2\, dx < \epsilon.$$
Let
$$\gamma_n = \int_0^\infty \psi_n(x) f_0(x)\, dx.$$
Then
$$\int_0^\infty \Big\{f(x)-\sum_{n=0}^N \gamma_n \phi_n(x)\Big\}^2 dx = \int_0^\infty \{f(x)\}^2\, dx + \sum_{n=0}^N \gamma_n^2 - 2\sum_{n=0}^N c_n \gamma_n$$
$$\geqslant \int_0^\infty \{f(x)\}^2\, dx - \sum_{n=0}^N c_n^2,$$
and also
$$\leqslant 2\int_0^\infty \{f(x)-f_0(x)\}^2\, dx + 2\int_0^\infty \Big\{f_0(x)-\sum_{n=0}^N \gamma_n \psi_n(x)\Big\}^2 dx$$
$$\leqslant 2\epsilon+2\Big[\int_0^\infty \{f_0(x)\}^2\, dx - \sum_{n=0}^N \gamma_n^2\Big].$$
By what has been proved, the last bracket is less than ϵ if N is large enough. Hence
$$\int_0^\infty \{f(x)\}^2\, dx \leqslant \sum_{n=0}^N c_n^2+4\epsilon$$
if N is large enough. Hence
$$\int_0^\infty \{f(x)\}^2\, dx \leqslant \sum_{n=0}^\infty c_n^2.$$
Combining this with Lemma 2.10, the result follows.

It also follows that, if $g(x)$ is another function of L^2, with 'Fourier coefficients' d_n, then

$$\int_0^\infty f(x)g(x)\,dx = \tfrac{1}{4}\int_0^\infty \{f(x)+g(x)\}^2\,dx - \tfrac{1}{4}\int_0^\infty \{f(x)-g(x)\}^2\,dx$$
$$= \tfrac{1}{4}\sum_{n=0}^\infty (c_n+d_n)^2 - \tfrac{1}{4}\sum_{n=0}^\infty (c_n-d_n)^2$$
$$= \sum_{n=0}^\infty c_n d_n. \tag{2.12.3}$$

This formula is also true for complex functions $f(x)$, $g(x)$, by separation into real and imaginary parts.

2.13. Deduction of Theorem 2.7 (i) from Theorem 2.7 (ii). From the formula

$$(\lambda_n-\lambda)\int_0^b \psi(x,\lambda)\phi(x,\lambda_n)\,dx$$
$$= W_0\{\psi(x,\lambda),\phi(x,\lambda_n)\} - W_b\{\psi(x,\lambda),\phi(x,\lambda_n)\},$$

and (2.5.3), it follows that the second term on the right-hand side tends to 0 as $b\to\infty$. Hence we may take $f(x) = \psi_n(x)$ in (2.9.2), which gives

$$\int_0^\infty G(x,y,\lambda)\psi_n(y)\,dy = \frac{\psi_n(x)}{\lambda-\lambda_n}. \tag{2.13.1}$$

Also $G(x,y,\lambda)$ is $L^2(0\leqslant y<\infty)$, for a fixed x, and λ not real. Hence

$$\sum_{n=0}^\infty \left|\frac{\psi_n(x)}{\lambda-\lambda_n}\right|^2 \tag{2.13.2}$$

is convergent.

Now let $f(x)$ satisfy the conditions of Theorem 2.7 (i), and let

$$g(x) = \lambda f(x) - L\{f(x)\}.$$

If λ is not real

$$\int_0^\infty \psi(y,\lambda)L\{f(y)\}\,dy = \int_0^\infty f(y)L\{\psi(y,\lambda)\}\,dy = \lambda\int_0^\infty \psi(y,\lambda)f(y)\,dy.$$

Put $\lambda = \lambda_n+iv$, multiply by v, and make $v\to 0$. Using Lemma 2.4 as in § 2.5, we obtain

$$\int_0^\infty \psi_n(y)L\{f(y)\}\,dy = \lambda_n\int_0^\infty \psi_n(y)f(y)\,dy$$
$$= \lambda_n c_n.$$

Hence, if d_n is the 'Fourier coefficient' of $g(x)$,

$$d_n = (\lambda-\lambda_n)c_n, \tag{2.13.3}$$

and

$$\sum |\lambda-\lambda_n|^2 c_n^2 \tag{2.13.4}$$

is convergent.

From (2.9.2), and (2.12.3) with $f(y)$ replaced by $G(x,y,\lambda)$, it follows that

$$\begin{aligned} f(x) &= \int\limits_0^\infty G(x,y,\lambda)g(y)\,dy \\ &= \sum_{n=0}^\infty \frac{\psi_n(x)}{\lambda-\lambda_n}(\lambda-\lambda_n)c_n \\ &= \sum_{n=0}^\infty c_n\psi_n(x), \end{aligned}$$

the required result. The absolute convergence of the series follows from that of (2.13.2) and (2.13.4).

2.14. To prove Theorem 2.7 (i) directly, the following lemma is required.

LEMMA 2.14. *If $f(x)$ is $L^2(0,\infty)$, x is fixed, and $v \neq 0$,*

$$\Phi(x,\lambda) = O(|\lambda|^{\frac{1}{4}}|v|^{-1}).$$

The formula

$$\int\limits_x^\xi (\xi-y)^2(y-x)\Phi''(y)\,dy = (\xi-x)^2\Phi(x)+\int\limits_x^\xi (6y-2x-4\xi)\Phi(y)\,dy$$

is at once verified by integration by parts. The left-hand side is equal to

$$\int\limits_x^\xi (\xi-y)^2(y-x)\{q(y)\Phi(y)-\lambda\Phi(y)+f(y)\}\,dy.$$

Writing

$$C = \left[\int\limits_0^\infty \{f(x)\}^2\,dx\right]^{\frac{1}{2}}$$

we have

$$\begin{aligned} \left|\int\limits_x^\xi (\xi-y)^2(y-x)\Phi(y)\,dy\right| &\leqslant (\xi-x)^3\int\limits_x^\xi |\Phi(y)|\,dy \\ &\leqslant (\xi-x)^3\left\{\int\limits_x^\xi dy\int\limits_x^\xi |\Phi(y)|^2\,dy\right\}^{\frac{1}{2}} \leqslant C(\xi-x)^{\frac{7}{2}}/|v| \end{aligned}$$

by Lemma 2.8. Similarly

$$\left|\int\limits_x^\xi (\xi-y)^2(y-x)q(y)\Phi(y)\,dy\right| \leqslant C(\xi-x)^{7/2}\max_{y\leqslant\xi}|q(y)|/|v|,$$

$$\left|\int\limits_x^\xi (\xi-y)^2(y-x)f(y)\,dy\right| \leqslant C(\xi-x)^{7/2},$$

$$\left|\int\limits_x^\xi (6y-2x-4\xi)\Phi(y)\,dy\right| \leqslant 4C(\xi-x)^{3/2}/|v|.$$

Hence $$|\Phi(x)| \leqslant \frac{C(\xi-x)^{3/2}}{|v|}\Big(|\lambda|+\max_{y\leqslant\xi}|q(y)|+|v|\Big)\frac{4C}{|v|(\xi-x)^{1/2}}.$$

The result follows on taking $\xi = x+|\lambda|^{-1/2}$.

2.15. Proof of Theorem 2.7 (i). The absolute convergence of the series (2.7.3) is first proved as in § 2.13. This depends on Lemma 2.10 only, not on Theorem 2.7 (ii). Now

$$\int\limits_{C(R)} \Phi(x,\lambda)\,d\lambda = 2\pi i \sum_{-R<\lambda_n<R} c_n\psi_n(x), \tag{2.15.1}$$

where $C(R)$ is the same contour as in § 2.12. By (2.9.1), and Lemma 2.14 applied to Φ^*,

$$\Phi(x,\lambda) = \frac{f(x)}{\lambda}+O\left(\frac{1}{|\lambda|^{3/4}|v|}\right). \tag{2.15.2}$$

The integral of the last term round the semicircles tends to zero, as in the case of the corresponding term in § 2.12. Hence the contributions of the semicircles to (2.15.1) together tend to $2\pi i f(x)$ as $R\to\infty$.

Hence finally it is a question of proving that

$$\lim_{R\to\infty}\int\limits_{R-i}^{R+i} \Phi(x,\lambda)\,d\lambda \to 0$$

(and similarly for $-R$). Let

$$\Omega(\lambda) = \Phi(x,\lambda)-\frac{f(x)}{\lambda}-\sum_{R-1\leqslant\lambda_n\leqslant R+1}\frac{c_n\psi_n(x)}{\lambda-\lambda_n}.$$

Then $\Omega(\lambda)$ is regular for $R-1\leqslant\lambda\leqslant R+1$, and

$$\Omega(\lambda) = O\left(\frac{1}{|\lambda^{3/4}v|}\right)+O\left(\frac{1}{|v|}\sum_{R-1\leqslant\lambda_n\leqslant R+1}|c_n\psi_n(x)|\right)=\frac{o(1)}{|v|}$$

as $u\to\infty$. Hence by Lemma 2.11

$$\Omega(R+iv) = o(1)$$

uniformly for $-1 \leqslant v \leqslant 1$. Hence

$$\int\limits_{R-i}^{R+i} \Omega(\lambda)\,d\lambda \to 0.$$

Also

$$\int\limits_{R-i}^{R+i} \sum_{R-1\leqslant\lambda_n\leqslant R+1} \frac{c_n\psi_n(x)}{\lambda-\lambda_n}\,d\lambda = \sum_{R-1\leqslant\lambda_n\leqslant R+1} c_n\psi_n(x)\int\limits_{R-i}^{R+i}\frac{d\lambda}{\lambda-\lambda_n}$$

$$= O\Big\{\sum_{R-1\leqslant\lambda_n\leqslant R+1} |c_n\psi_n(x)|\Big\} \to 0.$$

This completes the proof.

2.16. Deduction of Theorem 2.7 (ii) from Theorem 2.7 (i). Let $f(x)$ satisfy the conditions of Theorem 2.7 (i), and let

$$f_N(x) = \sum_{n=0}^{N} c_n\psi_n(x).$$

Then $f_N(x) \to f(x)$ uniformly over any finite interval, and

$$\int\limits_0^\infty \{f_N(x)\}^2\,dx = \sum_{n=0}^{N} c_n^2 \leqslant \int\limits_0^\infty \{f(x)\}^2\,dx.$$

Hence by Lemma 2.4

$$\lim_{N\to\infty}\int\limits_0^\infty f_N(x)f(x)\,dx = \int\limits_0^\infty \{f(x)\}^2\,dx,$$

i.e.
$$\lim_{N\to\infty}\sum_{n=0}^{N} c_n^2 = \int\limits_0^\infty \{f(x)\}^2\,dx.$$

This is the required result for the special class of functions. The general result then follows as in § 2.12.

2.17. THEOREM 2.17. *If $f(x)$ is $L^2(0,\infty)$, and λ is not equal to any of the λ_n,*

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n\psi_n(x)}{\lambda-\lambda_n}.$$

The series is absolutely convergent. This follows from Lemma 2.10 and the convergence of (2.13.2), if λ is not real; and the result, proved for any λ, clearly follows for all λ.

Now consider the integral

$$\int\limits_{C(R)} \frac{\Phi(x,z)}{z-\lambda}\,dz$$

and proceed as in § 2.15. By Lemma 2.14

$$\frac{\Phi(x,z)}{z-\lambda} = O\left(\frac{1}{|z|^{\frac{1}{2}}|\mathbf{I}(z)|}\right).$$

Hence the integrals round the semicircles tend to zero. The proof is then completed as in § 2.15.

2.18. The interval $(-\infty, \infty)$. Now consider the case where the interval extends to infinity, or has a singularity, at each end. We actually consider the case where the interval is $(-\infty, \infty)$, and $q(x)$ is continuous for $-\infty < x < \infty$.

Let $\phi(x,\lambda)$, $\theta(x,\lambda)$ be the solutions of $(L-\lambda)y = 0$ such that

$$\phi(0) = 0, \qquad \phi'(0) = -1,$$

$$\theta(0) = 1, \qquad \theta'(0) = 0.$$

Then $W(\phi,\theta) = 1$.

By the previous theory, there are functions $m_1(\lambda)$ and $m_2(\lambda)$, regular in the upper half-plane, such that

$$\psi_1(x,\lambda) = \theta(x,\lambda)+m_1(\lambda)\phi(x,\lambda) \tag{2.18.1}$$

is $L^2(-\infty, 0)$, and

$$\psi_2(x,\lambda) = \theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda) \tag{2.18.2}$$

is $L^2(0,\infty)$. Then

$$W(\psi_1,\psi_2) = m_1(\lambda)-m_2(\lambda).$$

As in § 2.5

$$\int\limits_{-\infty}^{0} |\psi_1(x,\lambda)|^2\,dx = \frac{\mathbf{I}(m_1)}{v}, \qquad \int\limits_{0}^{\infty} |\psi_2(x,\lambda)|^2\,dx = -\frac{\mathbf{I}(m_2)}{v}. \tag{2.18.3}$$

Hence $\mathbf{I}(m_1) > 0$, $\mathbf{I}(m_2) < 0$, for $v > 0$.

Let

$$G(x,y,\lambda) = \frac{\psi_2(x,\lambda)\psi_1(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y \leqslant x), \qquad \frac{\psi_1(x,\lambda)\psi_2(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y > x), \tag{2.18.4}$$

and

$$\Phi(x,\lambda) = \int\limits_{-\infty}^{\infty} G(x,y,\lambda)f(y)\,dy,$$

where f is the arbitrary function to be expanded. The expansion

will reduce to a series if both $m_1(\lambda)$ and $m_2(\lambda)$ are meromorphic functions. We have

$$\Phi(x,\lambda) = \frac{\theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda)}{m_1(\lambda)-m_2(\lambda)} \int\limits_{-\infty}^{x} \{\theta(y,\lambda)+m_1(\lambda)\phi(y,\lambda)\}f(y)\,dy +$$

$$+\frac{\theta(x,\lambda)+m_1(\lambda)\phi(x,\lambda)}{m_1(\lambda)-m_2(\lambda)} \int\limits_{x}^{\infty} \{\theta(y,\lambda)+m_2(\lambda)\phi(y,\lambda)\}f(y)\,dy. \tag{2.18.5}$$

There are three possibilities:

(i) At an eigenvalue λ_n, $m_1(\lambda_n) = m_2(\lambda_n) = \mu$,

$$m_1(\lambda)-m_2(\lambda) \sim (\lambda-\lambda_n)\nu.$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{1}{\nu}\{\theta(x,\lambda_n)+\mu\phi(x,\lambda_n)\} \int\limits_{-\infty}^{\infty} \{\theta(y,\lambda_n)+\mu\phi(y,\lambda_n)\}f(y)\,dy.$$

(ii) $m_1(\lambda)$ and $m_2(\lambda)$ both have simple zeros,

$$m_1(\lambda) \sim \mu_1(\lambda-\lambda_n), \qquad m_2(\lambda) \sim \mu_2(\lambda-\lambda_n).$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{1}{\mu_1-\mu_2}\theta(x,\lambda_n) \int\limits_{-\infty}^{\infty} \theta(y,\lambda_n)f(y)\,dy.$$

(iii) $m_1(\lambda)$ and $m_2(\lambda)$ both have simple poles,

$$m_1(\lambda) \sim \frac{\mu_1}{\lambda-\lambda_n}, \qquad m_2(\lambda) \sim \frac{\mu_2}{\lambda-\lambda_n}.$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{\mu_1\mu_2}{\mu_1-\mu_2}\phi(x,\lambda_n) \int\limits_{-\infty}^{\infty} \phi(y,\lambda_n)f(y)\,dy.$$

The theory of this case is much the same as that already considered.

Suppose in particular that $q(x)$ is an even function. Then $\phi(x,\lambda)$ is an even function of x, and $\theta(x,\lambda)$ is an odd function. It follows that, if $\theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda)$ is $L^2(0,\infty)$, $\theta(x,\lambda)-m_2(\lambda)\phi(x,\lambda)$ is $L^2(-\infty,0)$. Hence $m_1(\lambda) = -m_2(\lambda)$. All eigenvalues occur under (ii) or (iii), and each eigenfunction is either odd or even.

REFERENCES

Weyl (1), (2), (3), Titchmarsh (4), (5), (7), (9), Kemble (1).

III

THE GENERAL SINGULAR CASE

3.1. We now consider the same problem as in the last chapter, but we no longer make the hypothesis that $m(\lambda)$ is a meromorphic function. All that is known so far is that it is an analytic function of λ, regular in the upper half-plane, and that $\mathbf{I}m(\lambda) < 0$.

We proceed formally as follows. Defining $\Phi(x,\lambda)$ as before, it can be proved that

$$f(x) = \lim_{R\to\infty}\left\{-\frac{1}{i\pi}\int\limits_{-R+i\delta}^{R+i\delta}\Phi(x,\lambda)\,d\lambda\right\},$$

where $\delta > 0$. It is then a question of the behaviour of this integral as $\delta \to 0$. Now

$$\Phi(x,\lambda) = \psi(x,\lambda)\int\limits_0^x \phi(y,\lambda)f(y)\,dy + \phi(x,\lambda)\int\limits_x^\infty \psi(y,\lambda)f(y)\,dy$$

$$= \{\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)\}\int\limits_0^x \phi(y,\lambda)f(y)\,dy +$$

$$+\phi(x,\lambda)\int\limits_x^\infty \{\theta(y,\lambda)+m(\lambda)\phi(y,\lambda)\}f(y)\,dy.$$

We shall prove that

$$\lim_{\delta\to 0}\int\limits_0^\lambda \{-\mathbf{I}m(u+i\delta)\}\,du = k(\lambda),$$

where $k(\lambda)$ is a non-decreasing function of λ. Hence we obtain formally

$$\mathbf{I}\left[-\frac{1}{\pi}\int\limits_{-R+i\delta}^{R+i\delta}\Phi(x,\lambda)\,d\lambda\right]$$

$$= \mathbf{I}\left[-\frac{1}{\pi}\int\limits_{-R+i\delta}^{R+i\delta}\{\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)\}\,d\lambda\int\limits_0^x \phi(y,\lambda)f(y)\,dy\right]+$$

$$+\mathbf{I}\left[-\frac{1}{\pi}\int\limits_{-R+i\delta}^{R+i\delta}\phi(x,\lambda)\,d\lambda\int\limits_x^\infty \{\theta(y,\lambda)+m(\lambda)\phi(y,\lambda)\}f(y)\,dy\right]$$

$$\to \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\phi(x,\lambda)\,dk(\lambda)\int\limits_0^\infty \phi(y,\lambda)f(y)\,dy$$

as $R \to \infty$, $\delta \to 0$, since $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are real for real λ. Hence the expansion formula involves Stieltjes integrals, and is formally

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,\lambda)\, dk(\lambda) \int_{0}^{\infty} \phi(y,\lambda) f(y)\, dy. \tag{3.1.1}$$

If we write

$$g(\lambda) = \int_{0}^{\infty} \phi(y,\lambda) f(y)\, dy, \tag{3.1.2}$$

then (3.1.1) is

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,\lambda) g(\lambda)\, dk(\lambda). \tag{3.1.3}$$

Also

$$\int_{0}^{\infty} \{f(x)\}^2\, dx = \frac{1}{\pi}\int_{0}^{\infty} f(x)\, dx \int_{-\infty}^{\infty} \phi(x,\lambda) g(\lambda)\, dk(\lambda)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} g(\lambda)\, dk(\lambda) \int_{0}^{\infty} \phi(x,\lambda) f(x)\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \{g(\lambda)\}^2\, dk(\lambda). \tag{3.1.4}$$

This is the 'Parseval formula'.

Now consider the interval $(-\infty,\infty)$ instead of $(0,\infty)$. In this case $\Phi(x,\lambda)$ is given by the formulae of § 2.18. We obtain

$$\lim_{\delta\to 0} \int_{0}^{\lambda} -\mathbf{I}\, \frac{1}{m_1(u+i\delta)-m_2(u+i\delta)}\, du = \xi(\lambda), \tag{3.1.5}$$

$$\lim_{\delta\to 0} \int_{0}^{\lambda} -\mathbf{I}\, \frac{m_1(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\, du = \eta(\lambda), \tag{3.1.6}$$

$$\lim_{\delta\to 0} \int_{0}^{\lambda} -\mathbf{I}\, \frac{m_1(u+i\delta)m_2(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\, du = \zeta(\lambda), \tag{3.1.7}$$

where $\xi(\lambda)$ and $\zeta(\lambda)$ are non-decreasing functions of λ, and $\eta(\lambda)$ is of bounded variation. Now

$$\mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \Phi(x,\lambda)\, d\lambda\right]$$

$$= \mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \frac{\theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\, d\lambda \int_{-\infty}^{x} \{\theta(y,\lambda)+m_1(\lambda)\phi(y,\lambda)\} f(y)\, dy\right] +$$

$$+\mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \frac{\theta(x,\lambda)+m_1(\lambda)\phi(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\, d\lambda \int_{x}^{\infty} \{\theta(y,\lambda)+m_2(\lambda)\phi(y,\lambda)\} f(y)\, dy\right].$$

Proceeding as before, we obtain formally the expansion formula

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \theta(x,\lambda)\,d\xi(\lambda) \int\limits_{-\infty}^{\infty} \theta(y,\lambda)f(y)\,dy + $$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \theta(x,\lambda)\,d\eta(\lambda) \int\limits_{-\infty}^{\infty} \phi(y,\lambda)f(y)\,dy + $$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \phi(x,\lambda)\,d\eta(\lambda) \int\limits_{-\infty}^{\infty} \theta(y,\lambda)f(y)\,dy + $$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \phi(x,\lambda)\,d\zeta(\lambda) \int\limits_{-\infty}^{\infty} \phi(y,\lambda)f(y)\,dy. \qquad (3.1.8)$$

Writing

$$g(\lambda) = \int\limits_{-\infty}^{\infty} \theta(y,\lambda)f(y)\,dy, \qquad h(\lambda) = \int\limits_{-\infty}^{\infty} \phi(y,\lambda)f(y)\,dy, \qquad (3.1.9)$$

this is equivalent to

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \theta(x,\lambda)g(\lambda)\,d\xi(\lambda) + \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \theta(x,\lambda)h(\lambda)\,d\eta(\lambda) + $$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \phi(x,\lambda)g(\lambda)\,d\eta(\lambda) + \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \phi(x,\lambda)h(\lambda)\,d\zeta(\lambda). \qquad (3.1.10)$$

Multiplying by $f(x)$ and integrating again, we obtain the Parseval formula

$$\int\limits_{-\infty}^{\infty} \{f(x)\}^2\,dx = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \{g(\lambda)\}^2\,d\xi(\lambda) + \frac{2}{\pi}\int\limits_{-\infty}^{\infty} g(\lambda)h(\lambda)\,d\eta(\lambda) + $$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \{h(\lambda)\}^2\,d\zeta(\lambda). \qquad (3.1.11)$$

In many cases these formulae take a simpler form. Suppose, for example, that $m_1(\lambda)$ tends to a real limit as $\mathbf{I}(\lambda) \to 0$, so that

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda)$$

is $L^2(-\infty, 0)$. We then have formally

$$d\eta(u) = m_1(u)\,d\xi(u), \qquad d\zeta(u) = \{m_1(u)\}^2\,d\xi(u).$$

Hence (3.1.8) reduces to

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \psi_1(x,\lambda)\, d\xi(\lambda) \int\limits_{-\infty}^{\infty} \psi_1(y,\lambda) f(y)\, dy. \tag{3.1.12}$$

Again, consider the case where $q(x)$ is an even function. As in § 2.17, this involves $m_1(\lambda) = -m_2(\lambda)$. Hence

$$\frac{m_1(\lambda)}{m_1(\lambda)-m_2(\lambda)} = \frac{1}{2}$$

and $\eta(\lambda) = 0$. Hence the expansion formula is

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \theta(x,\lambda)\, d\xi(\lambda) \int\limits_{-\infty}^{\infty} \theta(y,\lambda) f(y)\, dy +$$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \phi(x,\lambda)\, d\zeta(\lambda) \int\limits_{-\infty}^{\infty} \phi(y,\lambda) f(y)\, dy. \tag{3.1.13}$$

3.2. To justify the above formulae, we first prove two lemmas.

LEMMA 3.2. *For any fixed u_1 and u_2*

$$\int\limits_{u_1}^{u_2} \mathbf{I}\{m(u+iv)\}\, du \tag{3.2.1}$$

is bounded as $v \to 0$.

Consider first the case of a finite interval $(0, b)$, with given boundary conditions at the ends. Let $\lambda_{n,b}$, $\psi_{n,b}(x)$ be the eigenvalues and eigenfunctions. Let $l(\lambda)$ be the function so denoted in §§ 2.1–2.2, and let $r_{n,b}$ be the residue of $l(\lambda)$ at $\lambda_{n,b}$. Since the formulae of Chapter II hold in particular in this case (or may be verified directly),

$$\int\limits_0^b \{\theta(x,\lambda)+l(\lambda)\phi(x,\lambda)\}\psi_{n,b}(x) = \frac{r_{n,b}^{\frac{1}{2}}}{\lambda-\lambda_{n,b}} \tag{3.2.2}$$

by (2.5.7). Hence the Parseval formula gives

$$\int\limits_0^b |\theta(x,\lambda)+l(\lambda)\phi(x,\lambda)|^2\, dx = \sum_{n=0}^{\infty} \frac{r_{n,b}}{(u-\lambda_{n,b})^2+v^2}. \tag{3.2.3}$$

Taking $\lambda = i$, the left-hand side is bounded as $b \to \infty$, by the analysis of § 2.1; e.g. in (2.1.6) $\mathbf{I}(l)$ is bounded as $b \to \infty$, by the property of the circles C_b, and $v = 1$. Hence

$$\sum_{n=0}^{\infty} \frac{r_{n,b}}{\lambda_{n,b}^2+1}$$

is bounded as $b \to \infty$.

Also, by (2.10.2) applied to the finite interval $(0,b)$, and (3.2.3),

$$-\mathbf{I}\{l(\lambda)\} = v\sum_{n=0}^{\infty}\frac{r_{n,b}}{(u-\lambda_{n,b})^2+v^2}.$$

Hence
$$\int_{u_1}^{u_2} -\mathbf{I}\{l(\lambda)\}\,du = \sum_{n=0}^{\infty} r_{n,b}\int_{u_1}^{u_2}\frac{v\,du}{(u-\lambda_{n,b})^2+v^2}.$$

Let $-N \leqslant u_1 < u_2 \leqslant N$, where $N \geqslant 1$. Then if $|\lambda_{n,b}| \geqslant 2N$, $|u-\lambda_{n,b}| \geqslant \frac{1}{2}\lambda_{n,b}$. Hence

$$\int_{u_1}^{u_2}\frac{v\,du}{(u-\lambda_{n,b})^2+v^2} \leqslant \int_{u_1}^{u_2}\frac{4\,du}{\lambda_{n,b}^2} \leqslant \frac{8N}{\lambda_{n,b}^2} \leqslant \frac{16N}{\lambda_{n,b}^2+1}.$$

If $|\lambda_{n,b}| < 2N$,

$$\int_{u_1}^{u_2}\frac{v\,du}{(u-\lambda_{n,b})^2+v^2} \leqslant \int_{-\infty}^{\infty}\frac{v\,du}{(u-\lambda_{n,b})^2+v^2} = \pi \leqslant \pi\frac{4N^2+1}{\lambda_{n,b}^2+1}.$$

Hence
$$\int_{u_1}^{u_2} -\mathbf{I}\{l(\lambda)\}\,du$$
is less than an upper bound independent of b and v. Since $l(\lambda) \to m(\lambda)$ as $b \to \infty$, the result follows.

3.3. LEMMA 3.3. *The function*

$$k(\lambda) = \lim_{\delta\to 0}\int_{\kappa}^{\lambda} -\mathbf{I}\{m(u+i\delta)\}\,du \tag{3.3.1}$$

exists for all κ and λ except possibly for values belonging to an enumerable set; $k(\lambda)$ is a non-decreasing function, and

$$\lim_{\delta\to 0}\int_{\kappa}^{\lambda} -\mathbf{I}\{\psi(x,u+i\delta)\}\,du = \int_{\kappa}^{\lambda}\phi(x,u)\,dk(u). \tag{3.3.2}$$

It follows from the above lemma and (2.5.2) that as $v \to 0$

$$\int_{u_1}^{u_2} du \int_{0}^{\infty} |\psi(x,\lambda)|^2\,dx = O\left(\frac{1}{v}\right) \tag{3.3.3}$$

for fixed u_1 and u_2. Now by (2.5.1), with $\lambda' = i$ (or any fixed value)

$$m(\lambda) = m(i)+(i-\lambda)\int_{0}^{\infty}\psi(x,\lambda)\psi(x,i)\,dx$$
$$= O\left\{\int_{0}^{\infty}|\psi(x,\lambda)|^2\,dx\right\}^{\frac{1}{2}}.$$

Hence

$$\int_{u_1}^{u_2} |m(\lambda)|\, du \leqslant \left\{(u_2-u_1)\int_{u_1}^{u_2} |m(\lambda)|^2\, du\right\}^{\frac{1}{2}} = O(v^{-\frac{1}{2}}). \tag{3.3.4}$$

Hence

$$\int_0^1 dv \int_{u_1}^{u_2} |m(\lambda)|\, du$$

is finite, and so

$$\int_0^1 |m(\lambda)|\, dv \tag{3.3.5}$$

exists for almost all u.

Let

$$M(\lambda) = \int_i^\lambda m(z)\, dz.$$

Then $M(\lambda)$ tends to a limit as $v \to 0$ for almost all u. Let u_0 be a value of u for which (3.3.5) exists. Then

$$M(\lambda) = \left(\int_i^{u_0+i} + \int_{u_0+i}^{u_0+iv} + \int_{u_0+iv}^{u+iv}\right) m(z)\, dz,$$

and the first term is constant, and the second term tends to a limit, as $v \to 0$. Hence the last term also tends to a limit. Taking imaginary parts, the existence of $k(\lambda)$ for almost all λ follows. Since $-\mathbf{I}m(\lambda) \geqslant 0$, $k(\lambda)$ is non-decreasing.

Actually the limit (3.3.1) exists wherever $k(\lambda)$ is continuous, i.e. except in an enumerable set. For at such a λ we can find η and η' such that

$$k(\lambda+\eta)-k(\lambda-\eta') < \epsilon$$

and such that the limit exists at $\lambda+\eta$ and $\lambda-\eta'$. Hence

$$\int_{\lambda-\eta'}^{\lambda} -\mathbf{I}\{m(u+i\delta)\}\, du \leqslant \int_{\lambda-\eta'}^{\lambda+\eta} -\mathbf{I}\{m(u+i\delta)\}\, du < \epsilon$$

for $\delta < \delta_0$. Hence

$$\left|\int_{u_0}^{\lambda} -\mathbf{I}\{m(u+i\delta)-m(u+i\delta')\}\, du\right|$$

$$= \left|\int_{u_0}^{\lambda-\eta'} -\mathbf{I}\{m(u+i\delta)-m(u+i\delta')\}\, du + \right.$$

$$\left. + \int_{\lambda-\eta'}^{\lambda} -\mathbf{I}\{m(u+i\delta)\}\, du - \int_{\lambda-\eta'}^{\lambda} -\mathbf{I}\{m(u+i\delta')\}\, du\right|$$

$$< 3\epsilon$$

if δ and δ' are sufficiently small. Hence the result.

Now suppose that k is continuous at κ and λ. We have

$$\int_{\kappa}^{\lambda} \mathbf{I}\psi(x,u+iv)\,du = \mathbf{I}\int_{\kappa}^{\lambda} \{\theta(x,u+iv)+m(u+iv)\phi(x,u+iv)\}\,du$$

$$= \int_{\kappa}^{\lambda} \mathbf{I}\{\theta(x,u+iv)\}\,du + \int_{\kappa}^{\lambda} \mathbf{R}\{m(u+iv)\}\mathbf{I}\{\phi(x,u+iv)\}\,du +$$

$$+\int_{\kappa}^{\lambda} \mathbf{I}m(u+iv)\mathbf{R}\phi(x,u+iv)\,du.$$

Since $\theta(x,u)$ and $\phi(x,u)$ are real for real u, $\mathbf{I}\theta(x,u+iv)$ and $\mathbf{I}\phi(x,u+iv)$ are $O(v)$, uniformly with respect to u over a finite interval. Hence the first two terms tend to 0 with v (using (3.3.4)). The third term is

$$[\mathbf{I}M(u+iv)\mathbf{R}\phi(x,u+iv)]_{\kappa}^{\lambda} - \int_{\kappa}^{\lambda} \mathbf{I}M(u+iv)\mathbf{R}\phi_u(x,u+iv)\,du$$

$$\to [\{k(u)+C\}\phi(x,u)]_{\kappa}^{\lambda} - \int_{\kappa}^{\lambda} \{k(u)+C\}\phi_u(x,u)\,du$$

$$= \int_{\kappa}^{\lambda} \phi(x,u)\,dk(u).$$

The process is justified since $\mathbf{I}M(\lambda)$ is bounded, by Lemma 3.2.

3.4. Let
$$\chi(x,\lambda) = \int_{\kappa}^{\lambda} \phi(x,u)\,dk(u). \tag{3.4.1}$$

Now
$$\int_{0}^{\infty} \left\{\int_{\kappa}^{\lambda} \mathbf{I}\psi(x,u+iv)\,du\right\}^2 dx < K, \tag{3.4.2}$$

where K is independent of v, if κ and λ are in a fixed interval. For by (3.2.2)

$$\int_{0}^{b} \psi_{n,b}(x)\,dx \int_{\kappa}^{\lambda} \mathbf{I}\psi_b(x,u+iv)\,du$$

$$= r_{n,b}^{\frac{1}{2}} \int_{\kappa}^{\lambda} \frac{-v\,du}{(u-\lambda_{n,b})^2+v^2} = O\left(\frac{r_{n,b}^{\frac{1}{2}}}{\lambda_{n,b}^2+1}\right).$$

Hence the Parseval theorem gives

$$\int_{0}^{b} \left\{\int_{\kappa}^{\lambda} \mathbf{I}\psi_b(x,u+iv)\,du\right\}^2 dx = O\sum_{n=0}^{\infty} \frac{r_{n,b}}{(\lambda_{n,b}^2+1)^2} = O(1)$$

by (3.2.4). The result follows on making $b\to\infty$. Making $v\to 0$, it follows that $\chi(x,\lambda)$ is $L^2(0,\infty)$.

Let $f(x)$ be $L^2(0,\infty)$. Then

$$g_1(\lambda) = \int_0^\infty \chi(y,\lambda)f(y)\,dy \tag{3.4.3}$$

exists, and is a bounded function of λ in any finite interval.

Now consider

$$\mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta}\Phi(x,\lambda)\,d\lambda\right]$$

$$= \mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta}\psi(x,\lambda)\,d\lambda\int_0^x \phi(y,\lambda)f(y)\,dy\right]+$$

$$+\mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta}\phi(x,\lambda)\,d\lambda\int_x^\infty \psi(y,\lambda)f(y)\,dy\right]. \tag{3.4.4}$$

Now

$$\mathbf{I}\{\psi(x,\lambda)\phi(y,\lambda)-\phi(x,\lambda)\psi(y,\lambda)\} = \mathbf{I}\{\theta(x,\lambda)\phi(y,\lambda)-\phi(x,\lambda)\theta(y,\lambda)\} = O(\delta)$$

as $\delta \to 0$ (for x and y in fixed intervals). Hence (3.4.4) is equal to

$$\mathbf{I}\left[-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta}\phi(x,\lambda)\,d\lambda\int_0^\infty \psi(y,\lambda)f(y)\,dy\right]+O(\delta)$$

as $\delta \to 0$ (R fixed). Next

$$\int_{-R+i\delta}^{R+i\delta}\mathbf{I}\phi(x,\lambda)\,d\lambda\int_0^\infty \mathbf{R}\psi(y,\lambda)f(y)\,dy$$

$$= O(\delta)\int_{-R}^{R}du\int_0^\infty |\psi(y,u+i\delta)f(y)|\,dy$$

$$= O(\delta)\left\{\int_{-R}^{R}\left(\int_0^\infty |\psi(y,u+i\delta)f(y)|\,dy\right)^2 du\right\}^{\frac{1}{2}}$$

$$= O(\delta)\left(\int_{-R}^{R}du\int_0^\infty |\psi(y,u+i\delta)|^2\,dy\right)^{\frac{1}{2}} = O(\delta^{\frac{1}{2}})$$

by (3.3.3). Similarly

$$\int_{-R}^{R}\{\mathbf{R}\phi(x,u+i\delta)-\phi(x,u)\}\,du\int_0^\infty \mathbf{I}\psi(y,\lambda)f(y)\,dy = O(\delta^{\frac{1}{2}}).$$

Hence (3.4.4) is equal to

$$-\frac{1}{\pi}\int_{-R}^{R}\phi(x,u)\,du\int_{0}^{\infty}\mathbf{I}\psi(y,u+i\delta)f(y)\,dy+O(\delta^{\frac{1}{2}}). \tag{3.4.5}$$

Now

$$\int_{\kappa}^{\lambda}du\int_{0}^{\infty}-\mathbf{I}\psi(y,u+i\delta)f(y)\,dy=\int_{0}^{\infty}f(y)\,dy\int_{\kappa}^{\lambda}-\mathbf{I}\psi(y,u+i\delta)\,du$$
$$\to g_1(\lambda)$$

as $\delta\to 0$, by Lemma 2.4, uniformly over a finite λ-range. Hence, integrating (3.4.5) by parts, we obtain

$$\frac{1}{\pi}\left[\phi(x,u)\int_{\kappa}^{u}du'\int_{0}^{\infty}-\mathbf{I}\psi(y,u'+i\delta)f(y)\,dy\right]_{-R}^{R}-$$
$$-\frac{1}{\pi}\int_{-R}^{R}\phi_u(x,u)\int_{\kappa}^{u}du'\int_{0}^{\infty}-\mathbf{I}\psi(y,u'+i\delta)f(y)\,dy$$
$$\to\frac{1}{\pi}[\phi(x,u)g_1(u)]_{-R}^{R}-\frac{1}{\pi}\int_{-R}^{R}\phi_u(x,u)g_1(u)\,du.$$

If g_1 is of bounded variation, this is equal to the Stieltjes integral

$$\frac{1}{\pi}\int_{-R}^{R}\phi(x,u)\,dg_1(u). \tag{3.4.6}$$

3.5. We now require the following theorem.

Let $\phi(\lambda)$ be an analytic function of $\lambda=u+iv$, regular for $v>0$. Let it be bounded on each line $v=$ constant, and let its maximum modulus on the line tend to 0 as $v\to\infty$. Let $\phi(\lambda)=p(u,v)+iq(u,v)$, and

$$\int_{-\infty}^{\infty}|p(u,v)|\,du\leqslant M\quad(v>0). \tag{3.5.1}$$

Then there is a function $\rho(t)$, of bounded variation in $(-\infty,\infty)$, such that

$$\phi(\lambda)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{d\rho(t)}{t-\lambda}\quad(v>0). \tag{3.5.2}$$

Also
$$\lim_{v\to 0}\int_{u_1}^{u_2}p(u,v)\,du=\rho(u_2)-\rho(u_1) \tag{3.5.3}$$

for all values of u_1 and u_2.

Integrating $\phi(z)/(z-\lambda)$ along the straight line $(-R+iy, R+iy)$ and round the semicircle above it, where $0 < y < v$, and making $R \to \infty$, we obtain

$$\phi(\lambda) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R+iy}^{R+iy}\frac{\phi(z)}{z-\lambda}\,dz. \tag{3.5.4}$$

Similarly

$$0 = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R+iy}^{R+iy}\frac{\phi(z)}{z-\lambda'}\,dz, \tag{3.5.5}$$

where $\lambda' = u+i(2y-v)$. Subtracting the conjugate of (3.5.5) from (3.5.4),

$$\phi(\lambda) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{p(x,y)}{x+iy-\lambda}\,dx. \tag{3.5.6}$$

From this and (3.5.1) it follows that

$$|\phi(\lambda)| \leqslant \frac{1}{\pi(v-y)}\int_{-\infty}^{\infty}|p(x,y)|\,dx \leqslant \frac{M}{\pi(v-y)},$$

and making $y \to 0$

$$|\phi(\lambda)| \leqslant \frac{M}{\pi v}. \tag{3.5.7}$$

Now let

$$\phi_1(\lambda) = \int_i^\lambda \phi(z)\,dz = p_1(u,v)+iq_1(u,v),$$

$$\phi_2(\lambda) = \int_i^\lambda \phi_1(z)\,dz = p_2(u,v)+iq_2(u,v).$$

By (3.5.7), $\phi_1(\lambda) = O\{\log(1/v)\}$ as $v \to 0$, uniformly over a finite u-interval. Hence $\phi_2(\lambda)$ tends to a limit as $v \to 0$, uniformly over a finite u-interval. This limit $\phi_2(u) = p_2(u,0)+iq_2(u,0)$ is thus a continuous function of u.

Now

$$\phi_1(\lambda) = -i\int_v^1 \phi(iy)\,dy + \int_0^u \phi(x+iv)\,dx,$$

and hence

$$p_1(u,v) = \int_v^1 q(0,y)\,dy + \int_0^u p(x,v)\,dx$$

$$= \int_v^1 q(0,y)\,dy + \chi(u,v)-\omega(u,v),$$

where

$$\chi(u,v) = \tfrac{1}{2}\int\limits_0^u \{|p(x,v)|+p(x,v)\}\,dx,$$

$$\omega(u,v) = \tfrac{1}{2}\int\limits_0^u \{|p(x,v)|-p(x,v)\}\,dx.$$

For each v, the functions χ and ω are positive non-decreasing functions of u, and $\chi \leqslant M$, $\omega \leqslant M$. Let

$$P(u,v,h) = \frac{p_2(u+h,v)-p_2(u,v)}{h}.$$

Then

$$P(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} p_1(x,v)\,dx$$

$$= \int\limits_v^1 q(0,y)\,dy + \chi_1(u,v,h)-\omega_1(u,v,h),$$

where

$$\chi_1(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} \chi(x,v)\,dx, \qquad \omega_1(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} \omega(x,v)\,dx.$$

For given u and v, the functions χ_1 and ω_1 are non-decreasing functions of h, and $|\chi_1| \leqslant M$, $|\omega_1| \leqslant M$. Hence if $(h_\nu, h_\nu+\delta_\nu)$ are any non-overlapping intervals,

$$\sum_\nu |P(u,v,h_\nu+\delta_\nu)-P(u,v,h_\nu)|$$

$$\leqslant \sum_\nu \{\chi_1(u,v,h_\nu+\delta_\nu)-\chi_1(u,v,h_\nu)\}+$$

$$+\sum_\nu \{\omega_1(u,v,h_\nu+\delta_\nu)-\omega_1(u,v,h_\nu)\} \leqslant 4M.$$

Making $v \to 0$, it follows that

$$\sum_\nu |P(u,0,h_\nu+\delta_\nu)-P(u,0,h_\nu)| \leqslant 4M.$$

Hence $P(u,0,h)$ is of bounded variation, and so tends to limits as $h \to \pm 0$. Thus $p_2(u,0)$ has everywhere right-hand and left-hand derivatives $p'_{2,+}(u,0)$ and $p'_{2,-}(u,0)$. Also

$$|P(u,v,h)-P(u,v,1)| \leqslant 4M.$$

Making $v \to 0$,

$$|P(u,0,h)| \leqslant 4M+|p_2(u+1,0)|+|p_2(u,0)|.$$

This is bounded in any finite interval. Hence $p_2(u,0)$ is absolutely

continuous, and so is the integral of its derivative, which exists almost everywhere.

Further, χ_1 and ω_1 are non-decreasing functions of u, for given h and v. Hence, if $(u_\nu, u_\nu+\delta_\nu)$ are non-overlapping intervals,

$$\sum_\nu |P(u_\nu+\delta_\nu, v, h)-P(u_\nu, v, h)| \leqslant 4M.$$

Making $v \to 0$, then $h \to \pm 0$, it follows that $p'_{2,+}(u,0)$ and $p'_{2,-}(u,0)$ are of bounded variation in $(-\infty, \infty)$. Let

$$\rho(u) = \tfrac{1}{2}\{p'_{2,+}(u+0,0)+p'_{2,+}(u-0,0)\}.$$

Integrating (3.5.6) by parts,

$$\phi(\lambda) = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{p_1(x,y)}{(x+iy-\lambda)^2}\,dx = \frac{2}{\pi i}\int\limits_{-\infty}^{\infty} \frac{p_2(x,y)}{(x+iy-\lambda)^3}\,dx,$$

the integrated terms vanishing since $p_1(x,y) = O(1)$, $p_2(x,y) = O(x)$ for fixed y. In the last formula we can make $y \to 0$, and obtain

$$\phi(\lambda) = \frac{2}{\pi i}\int\limits_{-\infty}^{\infty} \frac{p_2(x,0)}{(x-\lambda)^3}\,dx. \tag{3.5.8}$$

To justify this step, we observe that

$$\phi_1(\lambda) = \left(\int\limits_{i}^{u+i} + \int\limits_{u+i}^{u+iv}\right)\phi(z)\,dz = O(u)+O\left(\log\frac{1}{v}\right)$$

by (3.5.7). Hence

$$p_2(u,v)-p_2(u,0) = \mathbf{R}\int\limits_0^v i\phi_1(u+iy)\,dy = O\left\{v\left(u+\log\frac{1}{v}\right)\right\}. \tag{3.5.9}$$

Hence as $y \to 0$

$$\int\limits_{-\infty}^{\infty} \frac{p_2(x,y)-p_2(x,0)}{(x+iy-\lambda)^3}\,dx = O\int\limits_{-\infty}^{\infty} \frac{y(|x|+\log 1/y)}{|x|^3+v^3}\,dx = o(1);$$

and also $$\int\limits_{-\infty}^{\infty}\left\{\frac{1}{(x+iy-\lambda)^3}-\frac{1}{(x-\lambda)^3}\right\}p_2(x,0)\,dx = o(1),$$

since $p_2(x,0) = O(x)$, by (3.5.9) with v fixed.

Integrating (3.5.8) by parts,

$$\phi(\lambda) = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{p'_2(x,0)}{(x-\lambda)^2}\,dx = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{\rho(x)}{(x-\lambda)^2}\,dx, \tag{3.5.10}$$

and (3.5.2) follows on integrating by parts again.

Since (3.5.10) is uniformly convergent, on integrating over (u_1, u_2) and taking real parts we have

$$\int_{u_1}^{u_2} p(u,v)\,du = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{v}{(x-u_2)^2+v^2}-\frac{v}{(x-u_1)^2+v^2}\right\}\rho(x)\,dx.$$

(3.5.3) follows from this by the theory of Cauchy's singular integral.†

3.6. Let $f(x)$, the arbitrary function to be expanded, satisfy the same conditions as in Theorem 2.7 (i); viz. $f(x)$ and $Lf(x)$ are $L^2(0,\infty)$,

$$f(0)\cos\alpha+f'(0)\sin\alpha = 0,$$

and
$$W\{f(x),\psi(x,\lambda)\}\to 0$$

as $x\to\infty$ for every complex λ.

Using the notation of § 3.2, and denoting by $\Phi_b(x,\lambda)$ the function corresponding to $\Phi(x,\lambda)$ for the interval $(0,b)$, we have by (1.9.3)

$$\Phi_b(x,\lambda) = \sum_{n=0}^{\infty}\frac{c_{n,b}\psi_{n,b}(x)}{\lambda-\lambda_{n,b}},$$

where
$$c_{n,b} = \int_0^b \psi_{n,b}(x)f(x)\,dx.$$

Hence
$$-\mathbf{I}\Phi_b(x,\lambda) = \sum_{n=0}^{\infty}\frac{vc_{n,b}\psi_{n,b}(x)}{(u-\lambda_{n,b})^2+v^2}.$$

Hence, as in § 3.2,

$$\int_{u_1}^{u_2}|\mathbf{I}\Phi_b(x,\lambda)|\,du \leqslant \pi\sum_{n=0}^{\infty}|c_{n,b}\psi_{n,b}(x)|.$$

By (2.13.3), with $\lambda = i$,

$$d_{n,b} = (i-\lambda_{n,b})c_{n,b},$$

where $d_{n,b}$ is the Fourier coefficient of $if(x)-Lf(x)$. Now

$$\sum_{n=0}^{\infty}|d_{n,b}|^2 \leqslant \int_0^b |if(x)-Lf(x)|^2\,dx \leqslant \int_0^{\infty}|if(x)-Lf(x)|^2\,dx.$$

Also $\psi_{n,b}(x)/(i-\lambda_{n,b})$ is the Fourier coefficient of $G_b(x,y,\lambda)$. Hence

$$\sum_{n=0}^{\infty}\left|\frac{\psi_{n,b}(x)}{i-\lambda_{n,b}}\right|^2 \leqslant \int_0^b |G_b(x,y,i)|^2\,dy$$
$$= |\psi_b(x,i)|^2\int_0^x |\phi(y,i)|^2\,dy+|\phi(x,i)|^2\int_x^b |\psi_b(y,i)|^2\,dy.$$

† See E. C. Titchmarsh, *Fourier Integrals*, pp. 30–1.

Now $$\psi_b(x,i) = \psi(x,i)+\{l(i)-m(i)\}\phi(x,i),$$
where $\psi(x,i)$ is $L^2(0,\infty)$, and either $\phi(x,i)$ is L^2 and $l(i) \to m(i)$, or
$$l(i)-m(i) = O\left\{\int_0^b |\phi(x,i)|^2\,dx\right\}^{-1}$$
Hence, as $b \to \infty$,
$$\psi_b(x,i) \to \psi(x,i), \qquad \int_x^b |\psi_b(y,i)|^2\,dy \to \int_x^\infty |\psi(y,i)|^2\,dy,$$
$$\int_0^b |G_b(x,y,i)|^2\,dy \to \int_0^\infty |G(x,y)|^2\,dy.$$
Hence $$\int_{u_1}^{u_2} |\mathbf{I}\Phi_b(x,\lambda)|\,du \leqslant K,$$
where K is independent of u_1, u_2, v, and b. Making $b \to \infty$, then $u_1 \to -\infty$, $u_2 \to \infty$, it follows that
$$\int_{-\infty}^{\infty} |\mathbf{I}\Phi(x,\lambda)|\,du \leqslant K. \tag{3.6.1}$$
From (3.6.1) and (2.15.2) it follows that the conditions of the theorem of § 3.5 are satisfied by the function $i\Phi(x,\lambda)$. Hence
$$\Phi(x,\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\rho(t)}{\lambda-t}, \tag{3.6.2}$$
where $\rho(t) = \rho(t,x)$ is of bounded variation in $(-\infty,\infty)$.

We can now prove the following theorem.

THEOREM. *If $f(x)$ satisfies the above conditions and*
$$\chi(x,\lambda) = \int_\kappa^\lambda \phi(x,u)\,dk(u), \qquad g_1(\lambda) = \int_0^\infty \chi(y,\lambda)f(y)\,dy,$$
then $$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,u)\,dg_1(u). \tag{3.6.3}$$

Consider the integral $$\int \Phi(x,\lambda)\,d\lambda \tag{3.6.4}$$
taken round the rectangle with corners at $\pm R+i\delta$, $\pm R+i$. As in § 2.15,
$$\lim_{R\to\infty} \int_{-R+i}^{R+i} \Phi(x,\lambda)\,d\lambda = -\pi i f(x).$$

Also by (3.6.2)

$$\mathbf{R}\Phi(x,\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u-t}{(u-t)^2+v^2}\,d\rho(t).$$

Hence

$$\int_{\delta}^{1} \mathbf{R}\Phi(x, R+iv)\,dv = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\arctan\frac{1}{R-t} - \arctan\frac{\delta}{R-t}\right) d\rho(t)$$

$$= O\int_{-\infty}^{-\Delta} |d\rho(t)| + O\int_{-\Delta}^{\Delta} \frac{|d\rho(t)|}{R-t} + O\int_{\Delta}^{\infty} |d\rho(t)|$$

which tends to zero as $R \to \infty$ (choosing first Δ and then R) uniformly with respect to δ.

On taking the imaginary part of (3.6.4) and making first R sufficiently large, then δ sufficiently small, and using (3.4.6), we obtain (3.6.3).

It is also easily seen that, under the conditions of this section, $g_1(u)$ is of bounded variation in $(-\infty, \infty)$. If $\sin\alpha \neq 0$,

$$\Phi(0,\lambda) = \sin\alpha \int_{0}^{\infty} \psi(y,\lambda) f(y)\,dy,$$

and (3.6.1) with $x = 0$ gives

$$\int_{-\infty}^{\infty} \left| \int_{0}^{\infty} \mathbf{I}\psi(y,\lambda) f(y)\,dy \right| du \leqslant K.$$

Hence
$$\sum \int_{u_\nu}^{u_{\nu+1}} \left| \int_{0}^{\infty} \mathbf{I}\psi(y,\lambda) f(y)\,dy \right| du \leqslant K$$

for any set of u_ν. Hence

$$\sum \left| \int_{u_\nu}^{u_{\nu+1}} du \int_{0}^{\infty} \mathbf{I}\psi(y,\lambda) f(y)\,dy \right| \leqslant K,$$

and making $v \to 0$

$$\sum |g(u_{\nu+1}) - g(u_\nu)| \leqslant K.$$

If $\sin\alpha = 0$, we can argue similarly with $\Phi_x(0,\lambda)$.

3.7. The Parseval formula.

THEOREM 3.7. *Let $f(x)$ belong to $L^2(0,\infty)$. Then the sequence of functions*

$$g_n(\lambda) = \int_{0}^{n} \phi(y,\lambda) f(y)\,dy$$

converges in mean, with respect to $k(\lambda)$, over $(-\infty,\infty)$, to a limit $g(\lambda)$; i.e.

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\{g(\lambda)-g_n(\lambda)\}^2\,dk(\lambda)=0;$$

and

$$\int_0^{\infty}\{f(x)\}^2\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\{g(\lambda)\}^2\,dk(\lambda).$$

To prove this, we require the mean convergence theorem for Stieltjes integrals.

Consider first a finite interval (a,b) in which $k(x)$ is bounded and non-decreasing, and let there be a sequence of continuous functions $f_n(x)$ such that

$$\int_a^b\{f_m(x)-f_n(x)\}^2\,dk(x)\to 0 \tag{3.7.1}$$

as $m\to\infty$, $n\to\infty$.

The function $\xi=k(x)$ maps the interval $a\leqslant x\leqslant b$ on the interval $k(a)\leqslant x\leqslant k(b)$ in the following way. If $k(x)$ is continuous, to each x corresponds just one value of ξ. At a discontinuity, e.g. if $k(x_1-0)<k(x_1+0)$, the point x_1 corresponds to the whole interval $\{k(x_1-0),k(x_1+0)\}$ (open or closed as the case may be). If $k(x)$ is constant over an interval (x_1,x_2), every x in the interval corresponds to the same value of ξ. Thus an inverse function $x=x(\xi)$, of the same type as $k(x)$, is also defined.

If the function $f(x)$ is measurable† B (in particular, continuous), the function $f\{x(\xi)\}=F(\xi)$ is also measurable B. The Stieltjes integral of $f(x)$ with respect to $k(x)$ is then

$$\int_a^b f(x)\,dk(x)=\int_{k(a)}^{k(b)}F(\xi)\,d\xi.$$

Conversely, if $F(\xi)$ is measurable B, and $L\{k(a),k(b)\}$, then

$$\int_{k(a)}^{k(b)}F(\xi)\,d\xi=\int_a^b F\{k(x)\}\,dk(x).$$

We can therefore put (3.7.1) in the form

$$\int_{k(a)}^{k(b)}\{F_m(\xi)-F_n(\xi)\}^2\,d\xi\to 0.$$

† See E. W. Hobson, *The Theory of Functions of a Real Variable* (2nd ed.), vol. 1, §§ 132 and 445–8.

Hence there is a function $F(\xi)$ of $L^2\{k(a), k(b)\}$, such that

$$\int_{k(a)}^{k(b)} \{F(\xi)-F_n(\xi)\}^2\, d\xi \to 0.$$

The usual method of defining this function shows that in fact it is measurable B. Hence $f(x) = F\{k(x)\}$ is measurable B, and

$$\int_a^b \{f(x)-f_n(x)\}^2\, dk(x) \to 0.$$

The case of an infinite interval can be discussed similarly.

Now let $f(x)$ be any function of $L^2(0,\infty)$. Then we can define a sequence of functions $f_n(x)$, satisfying the conditions of the above theorem, and zero for $x > n$, such that

$$\lim_{n\to\infty} \int_0^\infty \{f(x)-f_n(x)\}^2\, dx = 0.$$

Let
$$h_n(\lambda) = \int_0^n \phi(y,\lambda) f_n(y)\, dy.$$

Then
$$\begin{aligned}\int_\kappa^\lambda h_n(u)\, dk(u) &= \int_\kappa^\lambda dk(u) \int_0^n \phi(y,u) f_n(y)\, dy \\ &= \int_0^n f_n(y)\, dy \int_\kappa^\lambda \phi(y,u)\, dk(u) \\ &= \int_0^n f_n(y)\chi(y,\lambda)\, dy \\ &= h_{1,n}(\lambda),\end{aligned}$$

say. The above theorem then gives

$$f_n(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,u)\, dh_{1,n}(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,u) h_n(u)\, dk(u).$$

This integral converges uniformly with respect to x over any finite interval. Hence the process leading to (3.1.4) is valid, so that

$$\int_0^\infty \{f_n(x)\}^2\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \{h_n(\lambda)\}^2\, dk(\lambda).$$

Similarly

$$\int_0^\infty \{f_m(x)-f_n(x)\}^2\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \{h_m(\lambda)-h_n(\lambda)\}^2\, dk(\lambda).$$

Hence the functions $h_n(\lambda)$ converge in mean square to a limit $g(\lambda)$, such that

$$\int_0^{\infty} \{f(x)\}^2\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \{g(\lambda)\}^2\,dk(\lambda).$$

If $G(\lambda)$ corresponds in the same way to $F(x)$, we shall clearly have similarly

$$\int_0^{\infty} \{f(x)-F(x)\}^2\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \{g(\lambda)-G(\lambda)\}^2\,dk(\lambda).$$

Taking $F(x) = f(x)$ $(x \leqslant n)$, 0 $(x > n)$, we have $G(\lambda) = g_n(\lambda)$. Hence

$$\int_{-\infty}^{\infty} \{g(\lambda)-g_n(\lambda)\}^2\,dk(\lambda) = \pi\int_n^{\infty} \{f(x)\}^2\,dx \to 0$$

as $n \to \infty$. Thus $g(\lambda)$ is also the mean square limit of $g_n(\lambda)$. This proves the theorem.

3.8. The interval $(-\infty,\infty)$. The rigorous discussion of this case is similar to that which has been given for the case $(0,\infty)$. We shall indicate briefly how it proceeds.

We define functions $\psi_1(x,\lambda)$, $\psi_2(x,\lambda)$, $G(x,y,\lambda)$ as in § 2.18. Let $G_{a,b}(x,y,\lambda)$, $\lambda_{n,a,b}$, $\psi_{n,a,b}(x)$ be the Green's function, eigenvalues, and eigenfunctions for the interval (a,b). Then (cf. (2.13.1))

$$\int_a^b G_{a,b}(x,y,\lambda)\psi_{n,a,b}(y)\,dy = \frac{\psi_{n,a,b}(x)}{\lambda-\lambda_{n,a,b}}. \tag{3.8.1}$$

Hence the Parseval theorem gives

$$\int_a^b |G_{a,b}(x,y,\lambda)|^2\,dy = \sum_{n=0}^{\infty} \frac{\psi_{n,a,b}^2(x)}{(u-\lambda_{n,a,b})^2+v^2}.$$

Hence, as in § 3.2,

$$\int_{u_1}^{u_2} du \int_a^b |G_{a,b}(x,y,\lambda)|^2\,dy = \sum_{n=0}^{\infty} \psi_{n,a,b}^2(x)\int_{u_1}^{u_2} \frac{du}{(u-\lambda_{n,a,b})^2+v^2}$$

$$= O\left\{\frac{1}{v}\sum_{n=0}^{\infty}\frac{\psi_{n,a,b}^2(x)}{\lambda_{n,a,b}^2+1}\right\} = O\left\{\frac{1}{v}\int_a^b |G_{a,b}(x,y,i)|^2\,dy\right\}$$

$$= O\left(\frac{1}{v}\right)$$

as $v \to 0$, uniformly with respect to a and b, x, u_1, and u_2 being fixed. Hence

$$\int_{u_1}^{u_2} du \int_{-\infty}^{\infty} |G(x,y,\lambda)|^2\, dy = O\left(\frac{1}{v}\right).$$

Now let $x = 0$; by (2.18.4)

$$G(0,y,\lambda) = \frac{\psi_1(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y \leqslant 0), \qquad \frac{\psi_2(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y > 0).$$

Hence, by (2.18.3),

$$\int_{u_1}^{u_2} \frac{\mathbf{I}\{m_1(\lambda)-m_2(\lambda)\}}{|m_1(\lambda)-m_2(\lambda)|^2}\, du = O(1),$$

i.e.
$$\int_{u_1}^{u_2} \mathbf{I}\left\{\frac{1}{m_1(\lambda)-m_2(\lambda)}\right\} du = O(1). \tag{3.8.2}$$

Also it is easily verified from (3.8.1) that

$$\int_a^b G'_{a,b}(x,y,\lambda)\psi_{n,a,b}(y)\, dy = \frac{\psi'_{n,a,b}(x)}{\lambda-\lambda_{n,a,b}},$$

where the dash denotes differentiation with respect to x. Arguing as before, we obtain

$$\int_{u_1}^{u_2} du \int_{-\infty}^{\infty} |G'(x,y,\lambda)|^2\, dy = O\left(\frac{1}{v}\right).$$

Now

$$G'(0,y,\lambda) = \frac{-m_2(\lambda)\psi_1(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y < 0), \qquad \frac{-m_1(\lambda)\psi_2(y,\lambda)}{m_1(\lambda)-m_2(\lambda)} \quad (y > 0).$$

Hence, by (2.18.3),

$$\int_{u_1}^{u_2} \frac{|m_2(\lambda)|^2\,\mathbf{I}\{m_1(\lambda)\} - |m_1(\lambda)|^2\,\mathbf{I}\{m_2(\lambda)\}}{|m_1(\lambda)-m_2(\lambda)|^2}\, du = O(1),$$

i.e.
$$\int_{u_1}^{u_2} \mathbf{I}\left\{\frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda)-m_2(\lambda)}\right\} du = O(1). \tag{3.8.3}$$

It is also easily verified that

$$\left\{\mathbf{I}\left(\frac{m_1}{m_1-m_2}\right)\right\}^2 \leqslant \mathbf{I}\left(\frac{1}{m_1-m_2}\right) \mathbf{I}\left(\frac{m_1 m_2}{m_1-m_2}\right).$$

Hence

$$\int_{u_1}^{u_2}\left|\mathbf{I}\left\{\frac{m_1(\lambda)}{m_1(\lambda)-m_2(\lambda)}\right\}\right|du \leqslant \int_{u_1}^{u_2}\left|\mathbf{I}\left\{\frac{1}{m_1(\lambda)-m_2(\lambda)}\right\}\mathbf{I}\left\{\frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda)-m_2(\lambda)}\right\}\right|^{\frac{1}{2}}du$$

$$\leqslant \left[\int_{u_1}^{u_2}\mathbf{I}\left\{\frac{1}{m_1(\lambda)-m_2(\lambda)}\right\}du\int_{u_1}^{u_2}\left|\mathbf{I}\left\{\frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda)-m_2(\lambda)}\right\}\right|du\right]^{\frac{1}{2}} = O(1).$$

The existence of the functions $\xi(\lambda)$, $\eta(\lambda)$, and $\zeta(\lambda)$ defined by (3.1.5)–(3.1.7) now follows as before.

Let

$$\chi_1(x,\lambda) = \int_{\kappa}^{\lambda}\theta(x,u)\,d\xi(u), \qquad \chi_2(x,\lambda) = \int_{\kappa}^{\lambda}\theta(x,u)\,d\eta(u),$$

$$\chi_3(x,\lambda) = \int_{\kappa}^{\lambda}\phi(x,u)\,d\eta(u), \qquad \chi_4(x,\lambda) = \int_{\kappa}^{\lambda}\phi(x,u)\,d\zeta(u),$$

and

$$g_\nu(\lambda) = \int_0^\infty \chi_\nu(y,\lambda)f(y)\,dy \quad (\nu = 1,\ 2,\ 3,\ 4).$$

Then the expansion formula is

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \theta(x,u)\,dg_1(u) + \phi(x,u)\,dg_2(u) + $$
$$+\theta(x,u)\,dg_3(u) + \phi(x,u)\,dg_4(u).$$

This can now be proved in the same way as before.

3.9. The spectrum. In the case of the x-interval $(0,\infty)$ the spectrum may be defined as the λ-set which is the complement of the set of points in the neighbourhood of which $k(\lambda)$ is constant. If $k(\lambda)$ is constant in an interval, so are $\chi(x,\lambda)$ and $g_1(\lambda)$ (§ 3.4). Such an interval contributes nothing to the representation formula (3.6.3) or the Parseval formula (3.1.4).

If $m(\lambda)$ is a meromorphic function, the spectrum is the set of its poles. This is called a point-spectrum. On the other hand, an interval throughout which $k(\lambda)$ increases steadily belongs to the continuous spectrum.

In the case of the x-interval $(-\infty,\infty)$, the spectrum is the complement of the set of points in the neighbourhood of which all the functions $\xi(\lambda)$, $\eta(\lambda)$, and $\zeta(\lambda)$ are constant.

REFERENCES

Weyl (1), (2), (3), Titchmarsh (6).

IV
EXAMPLES

4.1. In this chapter we consider a number of examples of the above theory.

The simplest case is the Fourier case, with $q(x) = 0$. If the interval is $(0,\infty)$, and the boundary condition at 0 is (2.1.4), then

$$\theta(x,\lambda) = \cos\alpha\cos(x\surd\lambda)+\lambda^{-\frac{1}{2}}\sin\alpha\sin(x\surd\lambda),$$
$$\phi(x,\lambda) = \sin\alpha\cos(x\surd\lambda)-\lambda^{-\frac{1}{2}}\cos\alpha\sin(x\surd\lambda).$$

The function $\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$ must be a multiple of $e^{ix\surd\lambda}$ if $\mathbf{I}(\lambda) > 0$, since $e^{-ix\surd\lambda}$ is not $L^2(0,\infty)$. Hence

$$m(\lambda) = \frac{\sin\alpha - i\surd\lambda\cos\alpha}{\cos\alpha+i\surd\lambda\sin\alpha}.$$

Hence

$$-\mathbf{I}m(\lambda) = \frac{\surd\lambda}{\cos^2\alpha+\lambda\sin^2\alpha} \quad (\lambda > 0), \qquad 0 \quad (\lambda < 0)$$

and (3.1.1) reduces to

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\surd\lambda\,\phi(x,\lambda)}{\cos^2\alpha+\lambda\sin^2\alpha}\,d\lambda\int_0^\infty \phi(y,\lambda)f(y)\,dy.$$

This of course can be verified under the usual conditions for Fourier's integral formula by direct consideration of the integrand. The general theorem of § 3.6 gives the formula in a comparatively indirect form. Consider, e.g., the case $\alpha = \frac{1}{2}\pi$. Then

$$\theta(x,\lambda) = \lambda^{-\frac{1}{2}}\sin(x\surd\lambda), \qquad \phi(x,\lambda) = \cos(x\surd\lambda),$$

$$k(\lambda) = \int_0^\lambda u^{-\frac{1}{2}}\,du \quad (\lambda > 0), \qquad 0 \quad (\lambda < 0),$$

$$\chi(x,\lambda) = \int_0^\lambda \cos(x\surd u)u^{-\frac{1}{2}}\,du = \frac{2\sin(x\surd\lambda)}{x},$$

$$g_1(\lambda) = \int_0^\infty \frac{2\sin(x\surd\lambda)}{x}f(x)\,dx.$$

Thus we obtain Fourier's cosine formula in the form

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos(x\surd u)\,d\left\{\int_0^\infty \frac{2\sin(x\surd u)}{x}f(x)\,dx\right\}.$$

In the case of the interval $(-\infty, \infty)$ we have

$$\theta(x,\lambda) = \cos(x\sqrt{\lambda}), \qquad \phi(x,\lambda) = -\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}},$$

$$m_1(\lambda) = i\sqrt{\lambda}, \qquad \psi_1(x,\lambda) = e^{-ix\sqrt{\lambda}},$$

$$m_2(\lambda) = -i\sqrt{\lambda}, \qquad \psi_2(x,\lambda) = e^{ix\sqrt{\lambda}},$$

$$\xi'(\lambda) = -\mathbf{I}\left\{\frac{1}{m_1(\lambda)-m_2(\lambda)}\right\} = -\mathbf{I}\left(\frac{1}{2i\sqrt{\lambda}}\right)$$

$$= \frac{1}{2\sqrt{\lambda}} \quad (\lambda > 0), \qquad 0 \quad (\lambda < 0),$$

$$\zeta'(\lambda) = -\mathbf{I}\left\{\frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda)-m_2(\lambda)}\right\} = -\mathbf{I}\left(\frac{-i\sqrt{\lambda}}{2}\right)$$

$$= \frac{\sqrt{\lambda}}{2} \quad (\lambda > 0), \qquad 0 \quad (\lambda < 0).$$

Hence (3.1.13) gives

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos(x\sqrt{\lambda})\,\frac{d\lambda}{2\sqrt{\lambda}}\int_{-\infty}^\infty \cos(y\sqrt{\lambda})f(y)\,dy +$$

$$+\frac{1}{\pi}\int_0^\infty \sin(x\sqrt{\lambda})\,\frac{d\lambda}{2\sqrt{\lambda}}\int_{-\infty}^\infty \sin(y\sqrt{\lambda})f(y)\,dy.$$

Putting $\lambda = s^2$, this gives the ordinary form of Fourier's formula.

4.2. The Hermite expansion. Let $q(x) = x^2$ $(-\infty < x < \infty)$. We have then to solve the equation

$$\frac{d^2y}{dx^2} - (x^2-\lambda)y = 0. \tag{4.2.1}$$

Putting $y = e^{-\frac{1}{2}x^2}y_1$, we obtain

$$\frac{d^2y_1}{dx^2} - 2x\frac{dy_1}{dx} + (\lambda-1)y_1 = 0.$$

A solution of this is

$$y_1 = \int_\infty^{(0+)} e^{-xz-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda-\frac{1}{2}}\,dz,$$

where $z^{-\frac{1}{2}\lambda-\frac{1}{2}} = \exp\{(-\frac{1}{2}\lambda-\frac{1}{2})\log z\}$, and $\log z$ is real at the beginning of the contour. For this gives

$$\frac{d^2y_1}{dx^2}-2x\frac{dy_1}{dx}+(\lambda-1)y_1 = \int\limits_{\infty}^{(0+)} e^{-xz-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda-\frac{1}{2}}(z^2+2xz+\lambda-1)\,dz$$

$$= -2\int\limits_{\infty}^{(0+)} \frac{d}{dz}(e^{-xz-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda+\frac{1}{2}})\,dz = 0.$$

A solution of (4.2.1) is therefore

$$\phi_0(x,\lambda) = e^{-\frac{1}{2}x^2}\int\limits_{\infty}^{(0+)} e^{-xz-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda-\frac{1}{2}}\,dz.$$

Since (4.2.1) is unaltered if x is replaced by $-x$, another solution is $\phi_0(-x,\lambda)$.

If we take the above contour so that $\mathbf{R}(z) \geqslant -1$ on it, then for a fixed λ

$$\phi_0(x) = O(e^{-\frac{1}{2}x^2+x})$$

as $x\to\infty$. It is also fairly easy to see that $\phi_0(-x)$ is large for large positive x; for the maximum of $xz-\frac{1}{4}z^2$ is at $z = 2x$, which indicates that the integral is roughly of the order of e^{x^2}. It follows that the functions $\psi_1(x,\lambda)$, $\psi_2(x,\lambda)$ defined in the general theory are multiples of $\phi_0(-x,\lambda)$, $\phi_0(x,\lambda)$ respectively. Let

$$W\{\phi_0(-x,\lambda),\phi_0(x,\lambda)\} = \omega(\lambda).$$

Then

$$G(x,y,\lambda) = \frac{\phi_0(x,\lambda)\phi_0(-y,\lambda)}{\omega(\lambda)} \quad (y \leqslant x),$$

$$= \frac{\phi_0(-x,\lambda)\phi_0(y,\lambda)}{\omega(\lambda)} \quad (y > x).$$

Now

$$\phi_0(0,\lambda) = \int\limits_{\infty}^{(0+)} e^{-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda-\frac{1}{2}}\,dz = (e^{-2\pi i(\frac{1}{2}\lambda+\frac{1}{2})}-1)\int\limits_{0}^{\infty} e^{-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda-\frac{1}{2}}\,dz$$

$$= -(1+e^{-\pi i\lambda})2^{-\frac{1}{2}\lambda-\frac{1}{2}}\Gamma(\tfrac{1}{4}-\tfrac{1}{4}\lambda)$$

for $\mathbf{R}(\lambda) < 1$, and so by analytic continuation for all λ. Similarly

$$\phi_0'(0,\lambda) = -\int\limits_{\infty}^{(0+)} e^{-\frac{1}{4}z^2}z^{-\frac{1}{2}\lambda+\frac{1}{2}}\,dz = (1+e^{-\pi i\lambda})2^{-\frac{1}{2}\lambda+\frac{1}{2}}\Gamma(\tfrac{3}{4}-\tfrac{1}{4}\lambda).$$

Hence

$$\omega(\lambda) = 2\phi_0(0,\lambda)\phi_0'(0,\lambda) = -(1+e^{-\pi i\lambda})^2\,2^{1-\lambda}\Gamma(\tfrac{1}{4}-\tfrac{1}{4}\lambda)\Gamma(\tfrac{3}{4}-\tfrac{1}{4}\lambda)$$

$$= -(1+e^{-\pi i\lambda})^2\,2^{\frac{3}{2}-\frac{1}{2}\lambda}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\lambda).$$

This has zeros at the points $\lambda = 2n+1$; they are double zeros for $n < 0$, single for $n \geqslant 0$ (on account of the poles of the Γ-function). But $\phi_0(x,\lambda)$ vanishes if $\lambda = 2n+1$ $(n < 0)$. Hence $G(x,y,\lambda)$ has simple poles at $\lambda = 2n+1$ $(n \geqslant 0)$. Now

$$\phi_0(x, 2n+1) = e^{-\frac{1}{4}x^2} \int\limits_{(0+)} e^{-xz-\frac{1}{4}z^2} z^{-n-1}\, dz.$$

Putting $z = 2(z'-x)$, this is

$$\frac{e^{\frac{1}{4}x^2}}{2^n} \int\limits_{(x+)} \frac{e^{-z'^2}}{(z'-x)^{n+1}}\, dz' = \frac{e^{\frac{1}{4}x^2}}{2^n . n!} 2\pi i \left(\frac{d}{dx}\right)^n e^{-x^2}$$

$$= \frac{2\pi i(-1)^n}{2^n . n!} e^{-\frac{1}{4}x^2} H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n. Also if

$$\lambda = 2n+1+\epsilon,$$

$$\omega(\lambda) = -(1-e^{-\pi i\epsilon})^2\, 2^{1-n-\frac{1}{2}\epsilon} \pi^{\frac{1}{2}} \Gamma(-n-\tfrac{1}{2}\epsilon)$$

$$\sim \pi^2 \epsilon^2\, 2^{1-n} \pi^{\frac{1}{2}} \frac{(-1)^n}{n!} \frac{1}{-\frac{1}{2}\epsilon} = \frac{(-1)^{n-1}\, 2^{2-n} \pi^{\frac{5}{2}}}{n!} \epsilon.$$

Also $\phi_0(-x, 2n+1) = A\phi_0(x, 2n+1)$, since the Wronskian vanishes, and $A = (-1)^n$, by considering $x \to 0$. Hence the function

$$\Phi(x,\lambda) = \int\limits_{-\infty}^{\infty} G(x,y,\lambda) f(y)\, dy$$

has the residues

$$\int\limits_{-\infty}^{\infty} \frac{n!}{2^{2-n}\pi^{\frac{5}{2}}} \frac{4\pi^2}{2^{2n}(n!)^2} e^{-\frac{1}{2}x^2} H_n(x) e^{-\frac{1}{2}y^2} H_n(y) f(y)\, dy$$

$$= \frac{e^{-\frac{1}{2}x^2} H_n(x)}{2^n n!\, \pi^{\frac{1}{2}}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} H_n(y) f(y)\, dy.$$

Hence the normalized eigenfunctions are

$$\frac{e^{-\frac{1}{2}x^2} H_n(x)}{2^{\frac{1}{2}n} (n!)^{\frac{1}{2}} \pi^{\frac{1}{4}}}.$$

4.3. The Legendre expansion. The more general second-order differential equation

$$a(X) \frac{d^2Y}{dX^2} + b(X) \frac{dY}{dX} + \{\lambda - c(X)\} Y = 0 \qquad (4.3.1)$$

can be reduced to the standard form as follows. Let

$$x = \int \frac{dX}{\sqrt{a(X)}}.$$

Then
$$\frac{d^2Y}{dx^2}+\beta(x)\frac{dY}{dx}+\{\lambda-\gamma(x)\}Y = 0,$$

where
$$\beta(x) = \frac{b(X)-\frac{1}{2}a'(X)}{\{a(X)\}^{\frac{1}{2}}}, \qquad \gamma(x) = c(X).$$

Putting $Y = r(x)y$, where

$$r(x) = e^{-\frac{1}{2}\int \beta(x)\,dx},$$

we obtain
$$\frac{d^2y}{dx^2}+\{\lambda-\tfrac{1}{4}\beta^2(x)-\tfrac{1}{2}\beta'(x)-\gamma(x)\}y = 0. \tag{4.3.2}$$

The general Legendre equation is

$$(1-X^2)\frac{d^2Y}{dX^2}-2X\frac{dY}{dX}+\left\{n(n+1)-\frac{m^2}{1-X^2}\right\}Y = 0. \tag{4.3.3}$$

Here $x = \sin^{-1}X$, $\beta(x) = -\tan x$, $r(x) = \sqrt{\sec x}$, and the equation reduces to

$$\frac{d^2y}{dx^2}+\{n(n+1)+\tfrac{1}{4}\tan^2x+\tfrac{1}{2}-m^2\sec^2x\}y = 0. \tag{4.3.4}$$

4.4. Lemma.

$$\int\limits_C \frac{\cos sz}{(\cos z)^{m+\frac{1}{2}}}\,dz = \frac{(-2)^m}{\frac{1}{2}\ldots(m-\frac{1}{2})}\,\frac{(2\pi)^{\frac{3}{2}}}{\Gamma(\frac{3}{4}-\frac{1}{2}m+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}m-\frac{1}{2}s)}$$

if C is a closed contour including $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ but excluding the other zeros of $\cos z$.

Denoting the integral by $f_m(s)$, we have

$$f_m(s) = \int\limits_C \frac{\cos(s-1)z\cos z-\sin(s-1)z\sin z}{(\cos z)^{m+\frac{1}{2}}}\,dz$$

$$= \int\limits_C \frac{\cos(s-1)z}{\cos^{m-\frac{1}{2}}z}\,dz+\frac{s-1}{m-\frac{1}{2}}\int\limits_C \cos(s-1)z.(\cos z)^{-m+\frac{1}{2}}\,dz$$

$$= \frac{m+s-\frac{3}{2}}{m-\frac{1}{2}}f_{m-1}(s-1)$$

$$= \frac{(m+s-\frac{3}{2})(m+s-\frac{3}{2}-2)\ldots(-m+s+\frac{1}{2})}{(m-\frac{1}{2})\ldots\frac{1}{2}}f_0(s-m).$$

Now

$$f_0(s-m) = \int\limits_C \frac{\cos(s-m)z}{\cos^{\frac{1}{2}}z}\,dz = 4\int\limits_0^{\frac{1}{2}\pi} \frac{\cos(s-m)z}{\cos^{\frac{1}{2}}z}\,dz$$

$$= \frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}{\Gamma(\frac{3}{4}+\frac{1}{2}s-\frac{1}{2}m)\Gamma(\frac{3}{4}-\frac{1}{2}s+\frac{1}{2}m)}$$

and

$$\Gamma(\tfrac{3}{4}-\tfrac{1}{2}s+\tfrac{1}{2}m) = (-\tfrac{1}{4}-\tfrac{1}{2}s+\tfrac{1}{2}m)...(\tfrac{3}{4}-\tfrac{1}{2}s-\tfrac{1}{2}m)\Gamma(\tfrac{3}{4}-\tfrac{1}{2}s-\tfrac{1}{2}m).$$

Hence the result follows.

4.5. The ordinary Legendre expansion. Taking $m = 0$ in (4.3.4), the corresponding basic equation is

$$\frac{d^2y}{dx^2}+(\lambda+\tfrac{1}{4}\tan^2x+\tfrac{1}{4})y = 0 \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi). \tag{4.5.1}$$

Putting $y = Y\sqrt{\cos x}$,

$$\frac{d^2Y}{dx^2}-\tan x\frac{dY}{dx}+(\lambda-\tfrac{1}{4})Y = 0.$$

Putting $x = \xi-\frac{1}{2}\pi$,

$$\frac{d^2Y}{d\xi^2}+\cot\xi\frac{dY}{d\xi}+(\lambda-\tfrac{1}{4})Y = 0. \tag{4.5.2}$$

A solution of this is

$$Y = \int\limits_C \frac{\cos sz}{(\cos z-\cos\xi)^{\frac{1}{2}}}\,dz,$$

where $\lambda = s^2$, and C is a closed contour surrounding ξ and $-\xi$, but excluding all other singularities. For it gives

$$\frac{dY}{d\xi} = -\int\limits_C \frac{\frac{1}{2}\sin\xi\cos sz}{(\cos z-\cos\xi)^{\frac{3}{2}}}\,dz,$$

$$\frac{d^2Y}{d\xi^2} = \int\limits_C \frac{\frac{3}{4}\sin^2\xi-\frac{1}{2}\cos\xi(\cos z-\cos\xi)}{(\cos z-\cos\xi)^{\frac{5}{2}}}\cos sz\,dz.$$

Hence

$$\frac{d^2Y}{d\xi^2}+\cot\xi\frac{dY}{d\xi}-\tfrac{1}{4}Y = \int\limits_C \frac{\frac{3}{4}-\frac{1}{2}\cos\xi\cos z-\frac{1}{4}\cos^2z}{(\cos z-\cos\xi)^{\frac{5}{2}}}\cos sz\,dz.$$

On the other hand, integrating by parts,

$$Y = -\int\limits_C \frac{\frac{1}{2}\sin z}{(\cos z-\cos\xi)^{\frac{3}{2}}}\frac{\sin sz}{s}\,dz$$

$$= -\int_C \frac{\frac{1}{2}\cos z(\cos z - \cos\xi) + \frac{3}{4}\sin^2 z}{(\cos z - \cos\xi)^{\frac{3}{2}}} \frac{\cos sz}{s^2}\,dz$$

$$= -\frac{1}{s^2}\int_C \frac{\frac{3}{4} - \frac{1}{4}\cos^2 z - \frac{1}{2}\cos z\cos\xi}{(\cos z - \cos\xi)^{\frac{3}{2}}}\cos sz\,dz.$$

Hence Y satisfies (4.5.2). A solution of (4.5.1) is therefore

$$\phi_0(x) = Y\cos^{\frac{1}{2}}x = 4\cos^{\frac{1}{2}}x\int_0^{\xi}\frac{\cos sz}{(\cos z - \cos\xi)^{\frac{1}{2}}}\,dz.$$

Clearly another solution is $\phi_0(-x)$.

If $x = \frac{1}{2}\pi - \delta$, $\delta \to 0$,

$$\phi_0(x) = 4\sin^{\frac{1}{2}}\delta\int_0^{\pi-\delta}\frac{\cos sz}{(\cos z + \cos\delta)^{\frac{1}{2}}}\,dz$$

$$= 4\sin^{\frac{1}{2}}\delta\int_\delta^{\pi}\frac{\cos s(\pi - \zeta)}{(\cos\delta - \cos\zeta)^{\frac{1}{2}}}\,d\zeta$$

$$\sim 4\delta^{\frac{1}{2}}\cos s\pi\int_\delta^{\pi}\frac{d\zeta}{\{\frac{1}{2}(\zeta^2 - \delta^2)\}^{\frac{1}{2}}}$$

$$= 4\sqrt{2}\,\delta^{\frac{1}{2}}\cos s\pi\int_1^{\pi/\delta}\frac{dt}{\sqrt{(t^2 - 1)}} \sim 4\sqrt{2}\cos s\pi\,\delta^{\frac{1}{2}}\log\frac{1}{\delta}$$

and

$$\phi_0(-x) = 4\sin^{\frac{1}{2}}\delta\int_0^{\delta}\frac{\cos sz}{(\cos z - \cos\delta)^{\frac{1}{2}}}\,dz = O\left\{\delta^{\frac{1}{2}}\int_0^{\delta}\frac{dz}{\sqrt{(\delta^2 - z^2)}}\right\} = O(\delta^{\frac{1}{2}}).$$

Hence all solutions of (4.5.1) are $L^2(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and we are in Weyl's limit-circle case. We shall here merely select the point on the limit-circle which leads to the ordinary Legendre expansion.

The function $m_2(\lambda)$ is a point on the limit-circle of the circles

$$l = -\frac{\theta(b)\cot\beta + \theta'(b)}{\phi(b)\cot\beta + \phi'(b)}$$

as $b \to \frac{1}{2}\pi$, where

$$\theta(x,\lambda) = \frac{\phi_0(-x) + \phi_0(x)}{2\phi_0(0)}, \qquad \phi(x,\lambda) = \frac{\phi_0(-x) - \phi_0(x)}{2\phi_0'(0)}.$$

Taking $\cot\beta = \infty$, we obtain

$$m_2(\lambda) = \frac{\phi_0'(0)}{\phi_0(0)}$$

and

$$\psi_2(x,\lambda) = \frac{\phi_0(-x)}{\phi_0(0)}.$$

Similarly $\qquad m_1(\lambda) = -\frac{\phi_0'(0)}{\phi_0(0)}, \qquad \psi_1(x,\lambda) = \frac{\phi_0(x)}{\phi_0(0)}.$

Hence for $y < x$

$$G(x,y,\lambda) = \frac{\dfrac{\phi_0(-x)}{\phi_0(0)}\dfrac{\phi_0(y)}{\phi_0(0)}}{-2\dfrac{\phi_0'(0)}{\phi_0(0)}} = -\frac{1}{2}\frac{\phi_0(-x)\phi_0(y)}{\phi_0(0)\phi_0'(0)}.$$

Now

$$\phi_0(0) = \int\limits_C \frac{\cos sz}{\cos^{\frac{1}{2}}z}\,dz = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}s)},$$

$$\phi_0'(0) = -\frac{1}{2}\int\limits_C \frac{\cos sz}{\cos^{\frac{3}{2}}z}\,dz = \frac{2^{\frac{3}{2}}\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{4}+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}s)}.$$

Hence

$$\begin{aligned}\phi_0(0)\phi_0'(0) &= \frac{16\pi^3}{\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}s)} \\ &= 16\pi\sin\pi(\tfrac{1}{4}-\tfrac{1}{2}s)\sin\pi(\tfrac{1}{4}+\tfrac{1}{2}s) \\ &= 8\pi\cos\pi s = 8\pi\cos\pi\surd\lambda.\end{aligned}$$

Hence the eigenvalues are

$$\lambda_n = (n+\tfrac{1}{2})^2 \quad (n = 0,\ 1,\ldots).$$

Now

$$\begin{aligned}\phi_0(x,\lambda_n) &= 4\cos^{\frac{1}{2}}x\int\limits_0^{\frac{1}{2}\pi+x}\frac{\cos(n+\frac{1}{2})z}{(\cos z+\sin x)^{\frac{1}{2}}}\,dz \\ &= 2\surd2\,\pi\cos^{\frac{1}{2}}x\,P_n(-\sin x) \\ &= (-1)^n 2\surd2\,\pi\cos^{\frac{1}{2}}x\,P_n(\sin x),\end{aligned}$$

$$\phi_0(-x,\lambda_n) = (-1)^n\phi_0(x,\lambda_n),$$

and, as $\lambda \to \lambda_n$, $\qquad \phi_0(0)\phi_0'(0) \sim \dfrac{(-1)^{n-1}4\pi^2(\lambda-\lambda_n)}{n+\frac{1}{2}}.$

Hence $\Phi(x,y,\lambda)$ has a pole at λ_n with residue

$$\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{n+\frac{1}{2}}{8\pi^2}8\pi^2\cos^{\frac{1}{2}}x\,P_n(\sin x)\cos^{\frac{1}{2}}y\,P_n(\sin y)f(y)\,dy$$

$$= (n+\tfrac{1}{2})\cos^{\frac{1}{2}}x\,P_n(\sin x)\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^{\frac{1}{2}}y\,P_n(\sin y)f(y)\,dy.$$

Thus the normalized eigenfunctions are

$$(n+\tfrac{1}{2})^{\frac{1}{2}}\cos^{\frac{1}{2}}x\,P_n(\sin x).$$

4.6. Next let

$$q(x) = -\tfrac{1}{4}\tan^2x+m^2\sec^2x-\tfrac{1}{4} \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi),$$

where m is a positive integer. The basic equation is then

$$\frac{d^2y}{dx^2}+(\lambda+\tfrac{1}{4}\tan^2x-m^2\sec^2x+\tfrac{1}{4})y = 0.$$

Putting $y = Y_1\sqrt{\cos x}$, $x = \xi-\tfrac{1}{2}\pi$,

$$\frac{d^2Y_1}{d\xi^2}+\cot\xi\frac{dY_1}{d\xi}+(\lambda-m^2\operatorname{cosec}^2\xi-\tfrac{1}{4})Y_1 = 0.$$

A solution of this is

$$Y_1 = \sin^{\frac{1}{2}m}\xi\left(\frac{d}{d\cos\xi}\right)^m Y,$$

where Y satisfies (4.5.2); i.e.

$$Y_1 = (-1)^m\tfrac{1}{2}...(m-\tfrac{1}{2})\sin^{\frac{1}{2}m}\xi\int\limits_C\frac{\cos sz}{(\cos z-\cos\xi)^{m+\frac{1}{2}}}\,dz.$$

Hence solutions are $\phi_0(x) = \cos^{\frac{1}{2}}x\,Y_1$ and $\phi_0(-x)$. Hence

$$\phi_0(0) = (-1)^m\tfrac{1}{2}...(m-\tfrac{1}{2})\int\limits_C\frac{\cos sz}{(\cos z)^{m+\frac{1}{2}}}\,dz$$

$$= \frac{2^{m+\frac{3}{2}}\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{4}-\frac{1}{2}m+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}m-\frac{1}{2}s)}$$

and $$\phi_0'(0) = (-1)^{m+1}\tfrac{1}{2}...(m+\tfrac{1}{2})\int\limits_C\frac{\cos sz}{(\cos z)^{m+\frac{3}{2}}}\,dz$$

$$= \frac{2^{m+\frac{5}{2}}\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}m-\frac{1}{2}s)}.$$

Hence

$$\phi_0(0)\phi_0'(0)$$

$$= \frac{2^{2m+4}\pi^3}{\Gamma(\frac{3}{4}-\frac{1}{2}m+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}m-\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}m-\frac{1}{2}s)}$$

$$= \frac{2^{2m+4}\pi^3\,.\,2^{-\frac{1}{2}-m+s}\,.\,2^{-\frac{1}{2}-m-s}}{\pi\Gamma(\frac{1}{2}-m+s)\Gamma(\frac{1}{2}-m-s)}$$

$$= \frac{8\pi^2}{\Gamma(\frac{1}{2}-m+s)\Gamma(\frac{1}{2}-m-s)}.$$

This vanishes if $\pm s = n-m+\frac{1}{2}$, i.e. $\lambda = (n-m+\frac{1}{2})^2$ $(n = 0, 1,...)$. Also

$$\phi_0\{x, (n-m+\tfrac{1}{2})^2\} = \sin^{\frac{1}{2}m+\frac{1}{2}}\xi \left(\frac{d}{d\cos\xi}\right)^m 2\sqrt{2}\,\pi P_{n-m}(\cos\xi).$$

This is 0 if $n < 2m$, and if $n \geqslant 2m$ it is

$$2\sqrt{2}\,\pi \sin^{\frac{1}{2}}\xi\, P_{n-m}^m(\cos\xi).$$

Putting $n = m+r$, it follows that the eigenvalues are $(r+\frac{1}{2})^2$, $r = m$, $m+1,...$.

If $s = n-m+\frac{1}{2}+\epsilon$,

$$\phi_0(0)\phi_0'(0) = \frac{8\pi^2}{\Gamma(n-2m+1+\epsilon)\Gamma(-\epsilon-n)} \sim \frac{8\pi^2}{(n-2m)!}(-1)^{n-1}n!\,\epsilon$$
$$\sim \frac{8\pi^2}{(n-2m)!}(-1)^{n-1}n!\,\frac{\lambda-(n-m+\frac{1}{2})^2}{2(n-m+\frac{1}{2})}.$$

Hence the residue is

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (-\tfrac{1}{2})\frac{(-1)^{n-1}8\pi^2\cos^{\frac{1}{2}}x\,P_{n-m}^m(\sin x)\cos^{\frac{1}{2}}y\,P_{n-m}^m(\sin y)}{8\pi^2(-1)^n n!/(n-2m)!\,2(n-m+\frac{1}{2})}f(y)\,dy$$

$$= \frac{(n-2m)!}{n!}(n-m+\tfrac{1}{2})\cos^{\frac{1}{2}}x\,P_{n-m}^m(\sin x)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^{\frac{1}{2}}y\,P_{n-m}^m(\sin y)f(y)\,dy.$$

The expansion may be written in the form

$$f(x) = \sum_{r=m}^{\infty}\frac{(r-m)!}{(r+m)!}(r+\tfrac{1}{2})\cos^{\frac{1}{2}}x\,P_r^m(\sin x)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^{\frac{1}{2}}y\,P_r^m(\sin y)f(y)\,dy.$$

4.7. If we apply the general expansion theorem to the above formulae, we obtain a proof of the Legendre expansion under rather restricted conditions. We shall now indicate briefly how to justify the expansion under the same conditions as an ordinary Fourier series.

We have in the notation of § 4.5

$$\phi_0(x) = \sin^{\frac{1}{2}}\xi \int \frac{e^{-isz}}{(\cos z-\cos\xi)^{\frac{1}{2}}}\,dz,$$

where the contour surrounds the points $z = \pm\xi$. This contour can be replaced by loops round these points, each loop coming from

$-i\infty$, passing over the point in the positive direction, and returning to $-i\infty$. In the neighbourhood of $z = \xi$

$$(\cos z - \cos\xi)^{\frac{1}{2}} = \left(2\sin\frac{\xi-z}{2}\sin\frac{\xi+z}{2}\right)^{\frac{1}{2}}$$
$$\sim (\xi-z)^{\frac{1}{2}}\sin^{\frac{1}{2}}\xi.$$

Hence the loop round this point contributes

$$\int \frac{e^{-isz}}{(\xi-z)^{\frac{1}{2}}}\,dz = \frac{1}{i}\int \frac{e^{-isz}}{(z-\xi)^{\frac{1}{2}}}\,dz = \frac{e^{-is\xi}}{i}\int \frac{e^{-isz'}}{z'^{\frac{1}{2}}}\,dz'$$
$$= \frac{e^{-is\xi}}{is^{\frac{1}{2}}}\,\frac{2\pi e^{-\frac{1}{4}i\pi}}{\Gamma(\frac{1}{2})}$$

with error
$$O\left(\left|\frac{e^{is\xi}}{s^{\frac{3}{2}}}\right|\right).$$

Treating the other loop similarly, we obtain altogether

$$\phi_0(x) = -\frac{4\pi^{\frac{1}{2}}}{s^{\frac{1}{2}}}\sin(s\xi+\tfrac{1}{4}\pi)\left\{1+O\left(\frac{1}{|s|}\right)\right\}, \tag{4.7.1}$$

provided that $|s\xi| > A$.

If $|s\xi| < A$, take the integral round a circle of fixed radius, and we obtain $\phi_0(x) = O(|\sin^{\frac{1}{2}}\xi|)$.

We have

$$\Phi(x,\lambda) = -\frac{1}{16\pi\cos\pi s}\left\{\phi_0(-x)\int_{-\frac{1}{2}\pi}^{x}\phi_0(y)f(y)\,dy + \right.$$
$$\left. +\phi_0(x)\int_{x}^{\frac{1}{2}\pi}\phi_0(-y)f(y)\,dy\right\}.$$

Substituting the leading term in (4.7.1) in the part with $y < x$, we obtain

$$-\frac{1}{s\cos\pi s}\sin(-sx+\tfrac{1}{2}\pi s+\tfrac{1}{4}\pi)\int_{-\frac{1}{2}\pi}^{x}\sin(sy+\tfrac{1}{2}\pi s+\tfrac{1}{4}\pi)f(y)\,dy$$
$$= \frac{1}{2s\cos\pi s}\int_{-\frac{1}{2}\pi}^{x}[\cos\{(y-x)s+\pi s+\tfrac{1}{2}\pi\}-\cos\{(x+y)s\}]f(y)\,dy.$$

For $\mathbf{I}(s) > 0$ this

$$\sim \frac{1}{2se^{-i\pi s}}\int_{-\frac{1}{2}\pi}^{x} e^{-i(\pi-x+y)s-\frac{1}{2}i\pi}f(y)\,dy = \frac{1}{2is}\int_{-\frac{1}{2}\pi}^{x} e^{i(x-y)s}f(y)\,dy,$$

and we proceed as in § 1.9. The result is that the expansion is valid under the same conditions as an ordinary Fourier series.

4.8. Fourier-Bessel series. This is the case where the interval is $(0,b)$, and $q(x) = (\nu^2-\frac{1}{4})/x^2$. Let $\lambda = s^2$. Solutions of

$$\frac{d^2y}{dx^2}+\left(s^2-\frac{\nu^2-\frac{1}{4}}{x^2}\right)y = 0$$

are $x^{\frac{1}{2}}J_\nu(xs)$, $x^{\frac{1}{2}}Y_\nu(xs)$. Let $\phi(x,\lambda)$, $\theta(x,\lambda)$ be the solutions such that

$$\phi(b) = 0, \qquad \phi'(b) = -1,$$
$$\theta(b) = 1, \qquad \theta'(b) = 0.$$

Then

$$\left.\begin{aligned}\phi(x,\lambda) &= \frac{\pi}{2}x^{\frac{1}{2}}b^{\frac{1}{2}}\{J_\nu(xs)Y_\nu(bs)-Y_\nu(xs)J_\nu(bs)\},\\ \theta(x,\lambda) &= \frac{\pi}{2}x^{\frac{1}{2}}b^{\frac{1}{2}}s\{J_\nu(xs)Y'_\nu(bs)-Y_\nu(xs)J'_\nu(bs)\}.\end{aligned}\right\} \tag{4.8.1}$$

If $\nu \geqslant 1$, this is an example of Weyl's 'limit-point' case. The only solution of $L^2(0,b)$ is $x^{\frac{1}{2}}J_\nu(xs)$, so that

$$\psi(x,\lambda) = \theta(x,\lambda)-\frac{J'_\nu(bs)}{J_\nu(bs)}s\phi(x,\lambda),$$

i.e.
$$m(\lambda) = -\sqrt{\lambda}\frac{J'_\nu(b\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})}. \tag{4.8.2}$$

The eigenvalues λ_n are the zeros of $J_\nu(b\sqrt{\lambda})$. Now

$$J_\nu(b\sqrt{\lambda}) = (\lambda-\lambda_n)\tfrac{1}{2}b\lambda_n^{-\frac{1}{2}}J'_\nu(b\sqrt{\lambda_n})+\dots.$$

Hence $m(\lambda)$ has the residue $r_n = -2\lambda_n/b$ (negative because the singularity is at the lower end). Also, if $\lambda_n = s_n^2$,

$$\phi(x,\lambda_n) = \frac{\pi}{2}x^{\frac{1}{2}}b^{\frac{1}{2}}J_\nu(xs_n)Y_\nu(bs_n) = -\frac{x^{\frac{1}{2}}b^{\frac{1}{2}}J_\nu(xs_n)}{s_n\,bJ'_\nu(bs_n)}.$$

Hence the normalized eigenfunctions are

$$\left(\frac{2\lambda_n}{b}\right)^{\frac{1}{2}}\frac{x^{\frac{1}{2}}J_\nu(xs_n)}{s_n\,b^{\frac{1}{2}}J'_\nu(bs_n)} = \frac{2^{\frac{1}{2}}x^{\frac{1}{2}}J_\nu(xs_n)}{bJ'_\nu(bs_n)},$$

and the Fourier-Bessel expansion is

$$f(x) = \frac{2}{b^2}\sum_{n=1}^{\infty}\frac{x^{\frac{1}{2}}J_\nu(xs_n)}{J'^2_\nu(bs_n)}\int_0^b y^{\frac{1}{2}}J_\nu(ys_n)f(y)\,dy. \tag{4.8.3}$$

If $0 \leqslant \nu < 1$, all solutions of the equation belong to $L^2(0,b)$, and we are in the limit-circle case. Consider first the case $0 < \nu < 1$. Since

$$Y_\nu(z) = \{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)\}\operatorname{cosec}\nu\pi$$

we have

$$\phi(x,\lambda) = -\frac{\pi x^{\frac{1}{2}} b^{\frac{1}{2}}}{2\sin\nu\pi}\{J_\nu(xs)J_{-\nu}(bs)-J_{-\nu}(xs)J_\nu(bs)\},$$

$$\theta(x,\lambda) = -\frac{\pi x^{\frac{1}{2}} b^{\frac{1}{2}} s}{2\sin\nu\pi}\{J_\nu(xs)J'_{-\nu}(bs)-J_{-\nu}(xs)J'_\nu(bs)\}.$$

The limit-circle is the limit of the circles

$$l = -\frac{\theta(a,\lambda)\cot\alpha+\theta'(a,\lambda)}{\phi(a,\lambda)\cot\alpha+\phi'(a,\lambda)} \tag{4.8.4}$$

as $a \to 0$. Now as $x \to 0$ (s fixed)

$$J_\nu(xs) = \frac{(xs)^\nu}{2^\nu\Gamma(1+\nu)}+O(x^{\nu+2}), \qquad J'_\nu(xs) = \frac{\nu x^{\nu-1}s^\nu}{2^\nu\Gamma(1+\nu)}+O(x^{\nu+1}).$$

Hence

$$\theta(a)\cot\alpha+\theta'(a) = -\frac{\pi b^{\frac{1}{2}} s}{2\sin\nu\pi}\left[\frac{s^\nu J'_{-\nu}(bs)}{2^\nu\Gamma(1+\nu)}\{a^{\frac{1}{2}+\nu}\cot\alpha+(\tfrac{1}{2}+\nu)a^{-\frac{1}{2}+\nu}\}-\right.$$

$$\left.-\frac{s^{-\nu}J'_\nu(bs)}{2^{-\nu}\Gamma(1-\nu)}\{a^{\frac{1}{2}-\nu}\cot\alpha+(\tfrac{1}{2}-\nu)a^{-\frac{1}{2}-\nu}\}+O(a^{\frac{5}{2}-\nu}|\cot\alpha|)+O(a^{\frac{3}{2}-\nu})\right].$$

Now let

$$\frac{a^{\frac{1}{2}-\nu}\cot\alpha+(\frac{1}{2}-\nu)a^{-\frac{1}{2}-\nu}}{2^{-\nu}\Gamma(1-\nu)} = c\,\frac{a^{\frac{1}{2}+\nu}\cot\alpha+(\frac{1}{2}+\nu)a^{-\frac{1}{2}+\nu}}{2^\nu\Gamma(1+\nu)},$$

where c is a constant. Then

$$\cot\alpha = \frac{2^\nu\Gamma(1+\nu)(\frac{1}{2}-\nu)a^{-\frac{1}{2}-\nu}-c\,2^{-\nu}\Gamma(1-\nu)(\frac{1}{2}+\nu)a^{-\frac{1}{2}+\nu}}{c\,2^{-\nu}\Gamma(1-\nu)a^{\frac{1}{2}+\nu}-2^\nu\Gamma(1+\nu)a^{\frac{1}{2}-\nu}} = O\left(\frac{1}{a}\right)$$

and

$$a^{\frac{1}{2}+\nu}\cot\alpha+(\tfrac{1}{2}+\nu)a^{-\frac{1}{2}+\nu} = \frac{-2^{\nu+1}\nu\Gamma(1+\nu)}{c\,2^{-\nu}\Gamma(1-\nu)a^{\frac{1}{2}+\nu}-2^\nu\Gamma(1+\nu)a^{\frac{1}{2}-\nu}} \asymp a^{\nu-\frac{1}{2}}.$$

Hence the O-terms in the above expression are negligible if $\frac{3}{2}-\nu > \nu-\frac{1}{2}$, i.e. $\nu < 1$. Treating the denominator in (4.8.4) similarly, (4.8.4) gives, when $a \to 0$,

$$m = -s\,\frac{cs^{-\nu}J'_\nu(bs)-s^\nu J'_{-\nu}(bs)}{cs^{-\nu}J_\nu(bs)-s^\nu J_{-\nu}(bs)}. \tag{4.8.5}$$

Since c may have any value, m describes the circle obtained by varying c, and so this is the limit circle. For each value of c, m is an even function of s, and so is a one-valued function of λ. Its only

singularities are poles, viz. the zeros of $cJ_\nu(b\sqrt{\lambda})-\lambda^\nu J_{-\nu}(b\sqrt{\lambda})$. Denoting these by λ_n, we have

$$\phi(x,\lambda_n) = -\frac{\pi x^{\frac{1}{2}}b^{\frac{1}{2}}}{2\sin\nu\pi}\{J_\nu(x\sqrt{\lambda_n})J_{-\nu}(b\sqrt{\lambda_n})-J_{-\nu}(x\sqrt{\lambda_n})J_\nu(b\sqrt{\lambda_n})\}$$

$$= \frac{\pi x^{\frac{1}{2}}b^{\frac{1}{2}}}{2\sin\nu\pi}J_\nu(b\sqrt{\lambda_n})\{c\lambda_n^{-\nu}J_\nu(x\sqrt{\lambda_n})-J_{-\nu}(x\sqrt{\lambda_n})\}.$$

The normalized eigenfunctions are $|r_n|^{\frac{1}{2}}\phi(x,\lambda_n)$, where the r_n are the residues of $m(\lambda)$ at the points λ_n. For $c=\infty$ the expansion is the ordinary Fourier-Bessel expansion of order ν, and for $c=0$ it is the expansion of order $-\nu$.

In the case $\nu=0$, we have

$$Y_0(xs) = \frac{2}{\pi}\left(\gamma+\log\frac{xs}{2}\right)\{1+O(x^2)\}, \qquad Y_0'(xs) = \frac{2}{\pi xs}+O(x|\log x|)$$

as $x\to 0$ (s fixed). Hence

$$\theta(a)\cot\alpha+\theta'(a) = \frac{\pi b^{\frac{1}{2}}s}{2}\left[\left\{Y_0'(bs)-J_0'(bs)\frac{2}{\pi}\left(\gamma+\log\frac{as}{2}\right)\right\}(a^{\frac{1}{2}}\cot\alpha+\tfrac{1}{2}a^{-\frac{1}{2}})-\right.$$

$$\left.-\frac{2}{\pi a^{\frac{1}{2}}}J_0'(bs)+O(a^{\frac{3}{2}}|\log a|\cot\alpha)+O(a^{\frac{1}{2}}|\log a|)\right]$$

$$= \frac{\pi b^{\frac{1}{2}}s}{2}\left[\left\{Y_0'(bs)-\frac{2}{\pi}J_0'(bs)\log s\right\}(a^{\frac{1}{2}}\cot\alpha+\tfrac{1}{2}a^{-\frac{1}{2}})-\right.$$

$$-J_0'(bs)\left\{\frac{2}{\pi}\left(\gamma+\log\frac{a}{2}\right)(a^{\frac{1}{2}}\cot\alpha+\tfrac{1}{2}a^{-\frac{1}{2}})+\frac{2}{\pi a^{\frac{1}{2}}}\right\}+$$

$$\left.+O(a^{\frac{3}{2}}|\log a|\cot\alpha)+O(a^{\frac{1}{2}}|\log a|)\right].$$

Taking

$$\frac{2}{\pi}\left(\gamma+\log\frac{a}{2}\right)(a^{\frac{1}{2}}\cot\alpha+\tfrac{1}{2}a^{-\frac{1}{2}})+\frac{2}{\pi a^{\frac{1}{2}}} = c(a^{\frac{1}{2}}\cot\alpha+\tfrac{1}{2}a^{-\frac{1}{2}})$$

and proceeding as before, we obtain

$$m = -s\frac{cJ_0'(bs)-\{Y_0'(bs)-(2/\pi)J_0'(bs)\log s\}}{cJ_0(bs)-\{Y_0(bs)-(2/\pi)J_0(bs)\log s\}}. \tag{4.8.6}$$

This is an even function of s (see Watson, *Theory of Bessel Functions*, 3.51 (3)), and a result similar to the previous one is obtained.

4.9. Direct discussion of Fourier-Bessel series. The general theory of Chapter II gives the Fourier-Bessel expansion under very restricted conditions. A direct discussion of the formulae involved gives the following more general result.

Theorem 4.9. *Let $f(y)$ be integrable over $(0, b)$, and let $\nu > -\frac{1}{2}$. Then if $0 < x < b$, the series (4.8.3) behaves as regards convergence in the same way as an ordinary Fourier series.*

Apply (1.7.5) to Bessel's equation, taking $\phi(b) = 0$, $\phi'(b) = 1$, instead of the conditions at $x = a$. We obtain

$$J_\nu(bs)Y_\nu(xs)-Y_\nu(bs)J_\nu(xs) = -\frac{2\sin\{s(b-x)\}}{\pi s}+O\left(\frac{e^{|t|(b-x)}}{|s|^2}\right).$$

Also, by the well-known asymptotic formula for Bessel functions,

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+O\left(\frac{|e^{-iz}|}{|z|^{\frac{3}{2}}}\right)$$

for $0 \leqslant \arg z \leqslant \frac{1}{2}\pi$, $|z| \geqslant 1$.

We have

$$\Phi(x,\lambda) = \tfrac{1}{2}\pi x^{\frac{1}{2}}\frac{J_\nu(bs)Y_\nu(xs)-Y_\nu(bs)J_\nu(xs)}{J_\nu(bs)}\int\limits_0^x y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy+$$

$$+\tfrac{1}{2}\pi x^{\frac{1}{2}}\frac{J_\nu(xs)}{J_\nu(bs)}\int\limits_x^b y^{\frac{1}{2}}\{J_\nu(bs)Y_\nu(ys)-Y_\nu(bs)J_\nu(ys)\}f(y)\,dy.$$

Let $0 < \delta < x$, and let

$$\Phi(x) = \Phi_1(x)+\Phi_2(x),$$

where

$$\Phi_1(x) = \frac{\pi}{2}x^{\frac{1}{2}}\frac{J_\nu(bs)Y_\nu(xs)-Y_\nu(bs)J_\nu(xs)}{J_\nu(bs)}\int\limits_0^\delta y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy,$$

and $\Phi_2(x)$ is the remainder. Now for $|s| > 1/\delta$

$$\int\limits_0^\delta y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy = \int\limits_0^{1/|s|}+\int\limits_{1/|s|}^{\delta}$$

$$= O\left\{\int\limits_0^{1/|s|} y^{\frac{1}{2}}|ys|^\nu|f(y)|\,dy\right\}+O\left\{\int\limits_{1/|s|}^{\delta} y^{\frac{1}{2}}\frac{e^{yt}}{|ys|^{\frac{1}{2}}}|f(y)|\,dy\right\}$$

$$= O\left\{|s|^{-\frac{1}{2}}\int\limits_0^{1/|s|}|ys|^{\nu+\frac{1}{2}}|f(y)|\,dy\right\}+O\left\{\frac{e^{\delta t}}{|s|^{\frac{1}{2}}}\int\limits_0^1|f(y)|\,dy\right\}$$

$$= O(|s|^{-\frac{1}{2}})+O\left(\frac{e^{\delta t}}{|s|^{\frac{1}{2}}}\right) = O\left(\frac{e^{\delta t}}{|s|^{\frac{1}{2}}}\right).$$

We now integrate round a quarter-square in the s-plane ($s = \sigma+it$) with sides on the lines

$$b\sigma = n\pi+\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi, \qquad bt = n\pi+\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi.$$

Here
$$\Phi_1(x) = O(|s|^{-\frac{1}{2}}e^{(\delta-x)t}),$$

and the integral of this is easily seen to be $O(n^{-\frac{1}{2}})$ (for fixed δ and x).

In $\Phi_2(x)$ we can use the asymptotic formulae throughout, and the argument proceeds as in § 1.9. We obtain, for example,

$$\frac{\pi}{2}x^{\frac{1}{2}}\frac{J_\nu(bs)Y_\nu(xs)-Y_\nu(bs)J_\nu(xs)}{J_\nu(bs)}\int\limits_\delta^x y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy$$

$$= -\int\limits_\delta^x \left\{\frac{\sin s(b-x)}{s}\,\frac{\cos(ys-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}{\cos(bs-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}+O\left(\frac{e^{t(y-x)}}{|s|^2}\right)\right\}f(y)\,dy.$$

On the contour considered

$$\frac{\cos(ys-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}{\cos(bs-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} = e^{-iy(s-b)}\left\{1+O\left(\frac{1}{|s|}\right)\right\}$$

and the leading term is then the same as in the case of an ordinary Fourier sine series. The result therefore follows.

Similar methods apply to the other Bessel-function formulae.

4.10. The Weber formula. Consider the case of Bessel functions with the interval (a,∞), where $a > 0$. Taking, e.g., $\alpha = 0$ in the boundary condition at a, we have

$$\phi(x,\lambda) = \frac{\pi}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\{J_\nu(xs)Y_\nu(as)-Y_\nu(xs)J_\nu(as)\},$$

$$\theta(x,\lambda) = \frac{\pi}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}s\{J_\nu(xs)Y'_\nu(as)-Y_\nu(xs)J'_\nu(as)\}.$$

The only solution which is small as $x\to\infty$, for $\mathbf{I}(s) > 0$, is

$$H_\nu^{(1)}(xs) = J_\nu(xs)+iY_\nu(xs).$$

It follows that

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

must be a multiple of this; hence

$$-sJ'_\nu(as)-m(\lambda)J_\nu(as) = i\{sY'_\nu(as)+m(\lambda)Y_\nu(as)\},$$

$$m(\lambda) = -s\frac{J'_\nu(as)+iY'_\nu(as)}{J_\nu(as)+iY_\nu(as)} = -s\frac{H_\nu^{(1)\prime}(as)}{H_\nu(as)}. \qquad (4.10.1)$$

Hence for $\lambda > 0$, i.e. s real,

$$-\mathbf{I}m(\lambda) = \mathbf{I}\left\{s\frac{J'_\nu(as)+iY'_\nu(as)}{J_\nu(as)+iY_\nu(as)}\right\}$$

$$= s\frac{J_\nu(as)Y'_\nu(as)-Y_\nu(as)J'_\nu(as)}{J_\nu^2(as)+Y_\nu^2(as)}$$

$$= \frac{2}{\pi a}\frac{1}{J_\nu^2(as)+Y_\nu^2(as)}.$$

Also
$$H_\nu^{(1)}(iz) = \frac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(z), \tag{4.10.2}$$

and $K_\nu(z)$ is real for real z. Hence

$$s\frac{H_\nu^{(1)\prime}(as)}{H_\nu^{(1)}(as)}$$

is real for purely imaginary s, i.e. for negative λ. The final result is therefore

$$f(x) = \int_0^\infty x^{\frac{1}{2}}\frac{J_\nu(xs)Y_\nu(as)-Y_\nu(xs)J_\nu(as)}{J_\nu^2(as)+Y_\nu^2(as)}\,s\,ds\,\times$$

$$\times\int_a^\infty y^{\frac{1}{2}}\{J_\nu(ys)Y_\nu(as)-Y_\nu(ys)J_\nu(as)\}f(y)\,dy. \tag{4.10.3}$$

If, instead of $\Phi(a) = 0$, it is assumed that

$$\Phi(a)\cos\alpha+\Phi'(a)\sin\alpha = 0,$$

we obtain the corresponding formula in which $J_\nu(as)$ and $Y_\nu(as)$ are replaced by

$$J_\nu(as)\left(\cos\alpha+\frac{\sin\alpha}{2a}\right)+J'_\nu(as)\sin\alpha s,$$

etc. For example, if
$$\tan\alpha = \frac{a}{2\nu-1},$$

then $J_\nu(as)$ and $Y_\nu(as)$ are replaced by $J_{\nu+1}(as)$ and $Y_{\nu+1}(as)$.

4.11. The Hankel formula. This is the Bessel-function case with interval $(0,\infty)$. Each end is now singular. Take $x = a$ as basic point instead of the $x = 0$ of the formulae of § 2.18. Then

$$\phi(x,\lambda) = \frac{\pi}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\{J_\nu(xs)Y_\nu(as)-Y_\nu(xs)J_\nu(as)\},$$

$$\theta(x,\lambda) = \frac{\pi}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}s\{J_\nu(xs)Y'_\nu(as)-Y_\nu(xs)J'_\nu(as)\}.$$

If $\nu > 1$, the solutions of $L^2(0,a)$ and $L^2(a,\infty)$ are $x^{\frac{1}{2}}J_\nu(xs)$ and $x^{\frac{1}{2}}H_\nu^{(1)}(xs)$. Hence

$$m_1(\lambda) = -s\frac{J'_\nu(as)}{J_\nu(as)}, \tag{4.11.1}$$

$$\psi_1(x,\lambda) = \frac{\pi}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}sJ_\nu(xs)\left\{Y'_\nu(as)-Y_\nu(as)\frac{J'_\nu(as)}{J_\nu(as)}\right\}$$
$$= \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\frac{J_\nu(xs)}{J_\nu(as)}.$$

As in § 4.10
$$m_2(\lambda) = -s\frac{H_\nu^{(1)\prime}(as)}{H_\nu^{(1)}(as)}. \tag{4.11.2}$$

Hence

$$-\mathbf{I}\frac{1}{m_1(\lambda)-m_2(\lambda)} = \mathbf{I}\frac{1}{s\left\{\dfrac{J'_\nu(as)}{J_\nu(as)}-\dfrac{H_\nu^{(1)\prime}(as)}{H_\nu^{(1)}(as)}\right\}}$$
$$= \mathbf{I}\left\{-\frac{\pi a}{2i}J_\nu(as)H_\nu^{(1)}(as)\right\} = \frac{\pi a}{2}J_\nu^2(as) \quad (s>0),$$
$$= 0 \quad (s = it,\ t>0).$$

Hence (3.1.12) gives

$$f(x) = \tfrac{1}{2}\int_0^\infty x^{\frac{1}{2}}J_\nu(x\surd\lambda)\,d\lambda\int_0^\infty y^{\frac{1}{2}}J_\nu(y\surd\lambda)f(y)\,dy$$
$$= \int_0^\infty x^{\frac{1}{2}}J_\nu(xs)s\,ds\int_0^\infty y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy. \tag{4.11.3}$$

Of course it is easily seen directly that

$$\Phi(x,s) = \frac{\pi}{2i}x^{\frac{1}{2}}H_\nu^{(1)}(xs)\int_0^x y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy +$$
$$+\frac{\pi}{2i}x^{\frac{1}{2}}J_\nu(xs)\int_x^\infty y^{\frac{1}{2}}H_\nu^{(1)}(ys)f(y)\,dy$$

and

$$-\mathbf{I}\Phi(x,s) = \frac{\pi}{2}x^{\frac{1}{2}}J_\nu(xs)\int_0^\infty y^{\frac{1}{2}}J_\nu(ys)f(y)\,dy \quad (s>0)$$
$$= 0 \quad (s=it,\ t>0).$$

This gives the result again.

Consider next the case $0<\nu<1$. Then $m_1(\lambda)$ is given by (4.8.5), with a instead of b, and $m_2(\lambda)$ by (4.11.2). This gives

$$\psi_1(x,\lambda) = \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\frac{cJ_\nu(xs)-s^{2\nu}J_{-\nu}(xs)}{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)}.$$

Also $$m_1(\lambda)-m_2(\lambda) = s\left\{\frac{H_\nu^{(1)\prime}(as)}{H_\nu^{(1)}(as)}-\frac{cJ'_\nu(as)-s^{2\nu}J'_{-\nu}(as)}{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)}\right\}.$$

Using the formula

$$H_\nu^{(1)}(z) = i\operatorname{cosec}\nu\pi\{e^{-i\nu\pi}J_\nu(z)-J_{-\nu}(z)\}$$

we obtain

$$m_1(\lambda)-m_2(\lambda) = \frac{2i}{\pi a}\,\frac{c-s^{2\nu}e^{-i\nu\pi}}{H_\nu^{(1)}(as)\{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)\}}.$$

Hence for $\lambda > 0$, i.e. s real and positive,

$$-\mathbf{I}\frac{1}{m_1(\lambda)-m_2(\lambda)} = \mathbf{R}\frac{\pi a}{2}\,\frac{H_\nu^{(1)}(as)\{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)\}}{c-s^{2\nu}e^{-i\nu\pi}}$$

$$= \frac{\pi a}{2}\{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)\}\frac{J_\nu(as)(c-s^{2\nu}\cos\nu\pi)+Y_\nu(as)s^{2\nu}\sin\nu\pi}{c^2-2cs^{2\nu}\cos 2\nu+s^{4\nu}}$$

$$= \frac{\pi a}{2}\,\frac{\{cJ_\nu(as)-s^{2\nu}J_{-\nu}(as)\}^2}{c^2-2cs^{2\nu}\cos\nu\pi+s^{4\nu}}.$$

For $\lambda < 0$, $s = it$, where t is real and positive. Now

$$H_\nu^{(1)}(iat) = \frac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(at)$$

and

$$cJ_\nu(iat)-(it)^{2\nu}J_{-\nu}(iat) = ce^{\frac{1}{2}\nu\pi i}I_\nu(at)-e^{\nu\pi i}t^{2\nu}e^{-\frac{1}{2}\nu\pi i}I_{-\nu}(at)$$

$$= e^{\frac{1}{2}\nu\pi i}\{cI_\nu(at)-t^{2\nu}I_{-\nu}(at)\}.$$

Hence $$-\frac{1}{m_1(\lambda)-m_2(\lambda)} = \frac{aK_\nu(at)\{cI_\nu(at)-t^{2\nu}I_{-\nu}(at)\}}{c-t^{2\nu}}.$$

This is real, and if $c < 0$ it is continuous. In this case the formula (3.1.12) gives

$$f(x) = \int_0^\infty \frac{x^{\frac{1}{2}}\{cJ_\nu(xs)-s^{2\nu}J_{-\nu}(xs)\}}{c^2-2cs^{2\nu}\cos\nu\pi+s^{4\nu}}s\,ds\int_0^\infty y^{\frac{1}{2}}\{cJ_\nu(ys)-s^{2\nu}J_{-\nu}(ys)\}f(y)\,dy. \tag{4.11.4}$$

If $c > 0$, there is also a pole at $t = c^{1/2\nu}$. Here

$$-\frac{1}{m_1(\lambda)-m_2(\lambda)} \sim -\frac{aK_\nu(at)c\{I_\nu(at)-I_{-\nu}(at)\}}{2\nu c^{(2\nu-1)/2\nu}(t-c^{1/2\nu})}$$

$$= ac^{1/2\nu}\frac{\sin\nu\pi}{\nu\pi}\,\frac{K_\nu^2(at)}{(t-c^{1/2\nu})}.$$

Also $$t-c^{1/2\nu} = \frac{t^2-c^{1/\nu}}{t+c^{1/2\nu}} \sim \frac{-\lambda-c^{1/\nu}}{2c^{1/2\nu}}.$$

Hence the residue of $1/\{m_1(\lambda)-m_2(\lambda)\}$ *qua* function of λ is

$$2ac^{1/\nu}\frac{\sin\nu\pi}{\nu\pi}K_\nu^2(ac^{1/2\nu}).$$

Also
$$\psi_1(x,-c^{1/\nu}) = \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\frac{K_\nu(xc^{1/2\nu})}{K_\nu(ac^{1/2\nu})}.$$

Hence we have to add to the above right-hand side the term

$$2c^{1/\nu}\frac{\sin\nu\pi}{\nu\pi}x^{\frac{1}{2}}K_\nu(xc^{1/2\nu})\int\limits_0^\infty y^{\frac{1}{2}}K_\nu(yc^{1/2\nu})f(y)\,dy. \tag{4.11.5}$$

In the case $\nu = 0$, $m_1(\lambda)$ is given by (4.8.6), with a instead of b, and $m_2(\lambda)$ by (4.11.2), with $\nu = 0$. Hence

$$\psi_1(x,\lambda) = \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\frac{cJ_0(xs)-Y_0(xs)+(2/\pi)J_0(xs)\log s}{cJ_0(as)-Y_0(as)+(2/\pi)J_0(as)\log s},$$

and

$$m_1(\lambda)-m_2(\lambda) = s\left\{\frac{H_0^{(1)\prime}(as)}{H_0^{(1)}(as)}-\frac{cJ_0'(as)-Y_0'(as)+(2/\pi)J_0'(as)\log s}{cJ_0(as)-Y_0(as)+(2/\pi)J_0(as)\log s}\right\}$$
$$= \frac{2}{\pi a}\frac{1+ic+i(2/\pi)\log s}{H_0^{(1)}(as)\{cJ_0(as)-Y_0(as)+(2/\pi)J_0(as)\log s\}}.$$

Hence for $\lambda > 0$, i.e. $s > 0$,

$$-\mathbf{I}\frac{1}{m_1(\lambda)-m_2(\lambda)} = \tfrac{1}{2}\pi a\frac{\{cJ_0(as)-Y_0(as)+(2/\pi)J_0(as)\log s\}^2}{\{c+(2/\pi)\log s\}^2+1}.$$

For $\lambda < 0$, i.e. $s = it$, $t > 0$, we obtain

$$-\frac{1}{m_1(\lambda)-m_2(\lambda)} = \frac{aK_0(at)\{cI_0(at)+(2/\pi)I_0(at)\log t+(2/\pi)K_0(at)\}}{c+(2/\pi)\log t}.$$

This has a pole at $t = e^{-\frac{1}{2}\pi c}$. As $t \to e^{-\frac{1}{2}\pi c}$ it

$$\sim \frac{aK_0^2(at)}{e^{\frac{1}{2}\pi c}(t-e^{-\frac{1}{2}\pi c})} \sim \frac{2aK_0^2(ae^{-\frac{1}{2}\pi c})}{-\lambda-e^{-\pi c}}.$$

Hence the complete formula is

$$f(x) = \int\limits_0^\infty \frac{x^{\frac{1}{2}}\{cJ_0(xs)-Y_0(xs)+(2/\pi)J_0(xs)\log s\}}{\{c+(2/\pi)\log s\}^2+1}s\,ds\,\times$$
$$\times\int\limits_0^\infty y^{\frac{1}{2}}\{cJ_0(ys)-Y_0(ys)+(2/\pi)J_0(ys)\log s\}f(y)\,dy\,+$$
$$+2x^{\frac{1}{2}}K_0(xe^{-\frac{1}{2}\pi c})\int\limits_0^\infty y^{\frac{1}{2}}K_0(ye^{-\frac{1}{2}\pi c})f(y)\,dy. \tag{4.11.6}$$

4.12. Further Bessel-function expansions. Let $q(x) = x$ $(0 < x < \infty)$. Solutions of

$$\frac{d^2y}{dx^2} - (x-\lambda)y = 0$$

are

$$\phi_0(x,\lambda) = (x-\lambda)^{\frac{1}{2}} I_{\frac{1}{3}}\{\tfrac{2}{3}(x-\lambda)^{\frac{3}{2}}\}, \qquad \psi_0(x,\lambda) = (x-\lambda)^{\frac{1}{2}} K_{\frac{1}{3}}\{\tfrac{2}{3}(x-\lambda)^{\frac{3}{2}}\},$$

where $(x-\lambda)^{\frac{1}{2}}$ is real and positive for λ real, $x > \lambda$.

(i) Let $\alpha = 0$ in the boundary condition at $x = 0$. Then

$$\phi(x,\lambda) = \frac{\phi_0(x,\lambda)\psi_0(0,\lambda) - \psi_0(x,\lambda)\phi_0(0,\lambda)}{\phi_0(0,\lambda)\psi_0'(0,\lambda) - \psi_0(0,\lambda)\phi_0'(0,\lambda)},$$

$$\theta(x,\lambda) = \frac{\phi_0(x,\lambda)\psi_0'(0,\lambda) - \psi_0(x,\lambda)\phi_0'(0,\lambda)}{\phi_0(0,\lambda)\psi_0'(0,\lambda) - \psi_0(0,\lambda)\phi_0'(0,\lambda)}.$$

Since $\phi_0(x,\lambda)$ is large as $x \to \infty$, and $\psi_0(x,\lambda)$ small, we must have

$$\psi(x,\lambda) = \theta(x,\lambda) - \frac{\psi_0'(0,\lambda)}{\psi_0(0,\lambda)}\phi(x,\lambda),$$

i.e.

$$m(\lambda) = -\frac{\psi_0'(0,\lambda)}{\psi_0(0,\lambda)}.$$

The eigenvalues λ_n are the zeros of $\psi_0(0,\lambda)$. Now

$$\psi_0(0,\lambda) = (\lambda-\lambda_n)\left\{\frac{\partial}{\partial\lambda}\psi_0(0,\lambda)\right\}_{\lambda_n} + \dots$$
$$= -(\lambda-\lambda_n)\psi_0'(0,\lambda_n) + \dots.$$

Hence $m(\lambda)$ has the residue 1 at each pole. Hence the normalized eigenfunctions are

$$\psi_n(x) = \phi(x,\lambda_n) = -\frac{\psi_0(x,\lambda_n)}{\psi_0'(0,\lambda_n)}.$$

To determine this in a real form, consider the value of $\psi_0(x,\lambda)$ when $x < \lambda$. As λ passes above x from the real axis on the left of it to the real axis on the right of it, $\arg(x-\lambda)$ goes from 0 to $-\pi$, and so $\arg(x-\lambda)^{\frac{3}{2}}$ goes from 0 to $-\frac{3}{2}\pi$. Now

$$K_{\frac{1}{3}}(z) = \tfrac{1}{2}\pi i e^{\frac{1}{6}\pi i} H_{\frac{1}{3}}^{(1)}(ze^{\frac{1}{2}\pi i}).$$

Hence

$$K_{\frac{1}{3}}(ze^{-\frac{3}{2}\pi i}) = \tfrac{1}{2}\pi i e^{\frac{1}{6}\pi i} H_{\frac{1}{3}}^{(1)}(ze^{-i\pi})$$
$$= \tfrac{1}{2}\pi i e^{\frac{1}{6}\pi i}\{H_{\frac{1}{3}}^{(1)}(z) + e^{-\frac{1}{3}\pi i} H_{\frac{1}{3}}^{(2)}(z)\}$$
$$= \frac{\pi e^{\frac{1}{6}\pi i}}{2\sin\frac{1}{3}\pi}[J_{-\frac{1}{3}}(z) - e^{-\frac{1}{3}\pi i} J_{\frac{1}{3}}(z) + e^{-\frac{1}{3}\pi i}\{e^{\frac{1}{3}\pi i} J_{\frac{1}{3}}(z) - J_{-\frac{1}{3}}(z)\}]$$
$$= \frac{\pi i}{\sqrt{3}}\{J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)\}.$$

Hence, if $x < \lambda$,

$$\psi_0(x,\lambda) = \frac{\pi}{\sqrt{3}}(\lambda-x)^{\frac{1}{2}}[J_{\frac{1}{3}}\{\tfrac{2}{3}(\lambda-x)^{\frac{3}{2}}\}+J_{-\frac{1}{3}}\{\tfrac{2}{3}(\lambda-x)^{\frac{3}{2}}\}].$$

Hence the λ_n are the zeros of

$$J_{\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}})+J_{-\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}}).$$

Also (from the recurrence formulae)

$$\psi_0'(x,\lambda) = \frac{\pi}{\sqrt{3}}(\lambda-x)[J_{\frac{2}{3}}\{\tfrac{2}{3}(\lambda-x)^{\frac{3}{2}}\}-J_{-\frac{2}{3}}\{\tfrac{2}{3}(\lambda-x)^{\frac{3}{2}}\}].$$

Hence the normalized eigenfunctions are

$$\psi_n(x) = \frac{-\sqrt{3}\,\psi_0(x,\lambda_n)}{\pi\lambda_n\{J_{\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}})-J_{-\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}})\}}.$$

(ii) Similarly, if $\alpha = \frac{1}{2}\pi$, we obtain

$$\psi_n(x) = \frac{\sqrt{3}\,\psi_0(x,\mu_n)}{\pi\mu_n\{J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})+J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})\}},$$

where the μ_n are the zeros of

$$J_{\frac{2}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}})-J_{-\frac{2}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}}).$$

The two expansions together make up the expansion corresponding to

$$q(x) = |x| \quad (-\infty < x < \infty).$$

4.13. Let $q(x) = -x$ $(0 \leqslant x < \infty)$, and let $\alpha = 0$ in the boundary condition at $x = 0$. Solutions of

$$\frac{d^2y}{dx^2}+(\lambda+x)y = 0$$

are $$(x+\lambda)^{\frac{1}{2}}J_{\frac{1}{3}}\{\tfrac{2}{3}(x+\lambda)^{\frac{3}{2}}\}, \qquad (x+\lambda)^{\frac{1}{2}}Y_{\frac{1}{3}}\{\tfrac{2}{3}(x+\lambda)^{\frac{3}{2}}\}$$

and the Wronskian of these is $3/\pi$. Writing for brevity

$$X = \tfrac{2}{3}(x+\lambda)^{\frac{3}{2}}, \qquad Y = \tfrac{2}{3}(y+\lambda)^{\frac{3}{2}}, \qquad Z = \tfrac{2}{3}\lambda^{\frac{3}{2}}$$

we have therefore

$$\phi(x,\lambda) = \frac{\pi}{3}\lambda^{\frac{1}{2}}(x+\lambda)^{\frac{1}{2}}\{J_{\frac{1}{3}}(X)Y_{\frac{1}{3}}(Z)-Y_{\frac{1}{3}}(X)J_{\frac{1}{3}}(Z)\},$$

$$\theta(x,\lambda) = \frac{\pi}{3}\lambda(x+\lambda)^{\frac{1}{2}}\{J_{\frac{1}{3}}(X)Y'_{\frac{1}{3}}(Z)-Y_{\frac{1}{3}}(X)J'_{\frac{1}{3}}(Z)\}.$$

Now $H_{\frac{1}{3}}^{(1)}(X)$ contains a factor

$$e^{iX} = \exp\left\{\frac{2i}{3}(x+\lambda)^{\frac{3}{2}}\right\} = \exp\left\{\frac{2i}{3}x^{\frac{3}{2}}\left(1+\frac{3}{2}\frac{\lambda}{x}+\ldots\right)\right\}$$

which is exponentially small as $x \to \infty$, if $\mathbf{I}(\lambda) > 0$. All other solutions of the equation are exponentially large, so that $\psi(x,\lambda)$ must be a multiple of $H_{\frac{1}{3}}^{(1)}(X)$. Hence

$$-\lambda^{\frac{1}{2}} J'_{\frac{1}{3}}(Z) - m(\lambda) J_{\frac{1}{3}}(Z) = i\{\lambda^{\frac{1}{2}} Y'_{\frac{1}{3}}(Z) + m(\lambda) Y_{\frac{1}{3}}(Z)\},$$

$$m(\lambda) = -\lambda^{\frac{1}{2}} \frac{J'_{\frac{1}{3}}(Z)+iY'_{\frac{1}{3}}(Z)}{J_{\frac{1}{3}}(Z)+iY_{\frac{1}{3}}(Z)} = -\lambda^{\frac{1}{2}} \frac{H_{\frac{1}{3}}^{(1)\prime}(Z)}{H_{\frac{1}{3}}^{(1)}(Z)}.$$

For $\lambda > 0$, i.e. Z real,

$$-\mathbf{I}m(\lambda) = \lambda^{\frac{1}{2}} \frac{J_{\frac{1}{3}}(Z) Y'_{\frac{1}{3}}(Z) - Y_{\frac{1}{3}}(Z) J'_{\frac{1}{3}}(Z)}{J^2_{\frac{1}{3}}(Z) + Y^2_{\frac{1}{3}}(Z)} = \frac{3}{\pi\lambda} \frac{1}{J^2_{\frac{1}{3}}(Z) + Y^2_{\frac{1}{3}}(Z)}.$$

For $\lambda < 0$, let $\lambda = \mu e^{i\pi}$. Then $Z = \frac{2}{3}\mu^{\frac{3}{2}} e^{\frac{3}{2}i\pi}$. Now

$$H_\nu^{(1)}(xe^{\frac{3}{2}i\pi}) = \frac{2}{\pi i} e^{-\frac{1}{2}\nu\pi i} K_\nu(xe^{i\pi})$$

$$= \frac{2}{\pi i} e^{-\frac{1}{2}\nu\pi i} \{e^{-\nu\pi i} K_\nu(x) - \pi i I_\nu(x)\}.$$

Hence

$$H_{\frac{1}{3}}^{(1)}(xe^{\frac{3}{2}i\pi}) = 2\left\{\frac{e^{-\frac{1}{2}\pi i}}{\pi i} K_{\frac{1}{3}}(x) - e^{-\frac{1}{6}\pi i} I_{\frac{1}{3}}(x)\right\}$$

$$= -2\left\{\frac{1}{\pi} K_{\frac{1}{3}}(x) + e^{-\frac{1}{6}\pi i} I_{\frac{1}{3}}(x)\right\}.$$

Hence
$$e^{\frac{3}{2}i\pi} \frac{H_{\frac{1}{3}}^{(1)\prime}(xe^{\frac{3}{2}i\pi})}{H_{\frac{1}{3}}^{(1)}(xe^{\frac{3}{2}i\pi})} = \frac{K'_{\frac{1}{3}}(x) + \pi e^{-\frac{1}{6}\pi i} I'_{\frac{1}{3}}(x)}{K_{\frac{1}{3}}(x) + \pi e^{-\frac{1}{6}\pi i} I_{\frac{1}{3}}(x)}.$$

Hence

$$-\mathbf{I}m(\lambda) = -\mathbf{I}\mu^{\frac{1}{2}} \frac{K'_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) + \pi e^{-\frac{1}{6}\pi i} I'_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}})}{K_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) + \pi e^{-\frac{1}{6}\pi i} I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}})}$$

$$= -\mu^{\frac{1}{2}} \frac{\frac{1}{2}\pi\{I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) K'_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) - I'_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) K_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}})\}}{\{K_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) + \pi e^{-\frac{1}{6}\pi i} I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}})\}\{K_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) + \pi e^{\frac{1}{6}\pi i} I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}})\}}$$

$$= -\frac{\mu^{\frac{1}{2}}\pi}{2}\left(-\frac{1}{\frac{2}{3}\mu^{\frac{3}{2}}}\right) \frac{1}{K^2_{\frac{1}{3}} + \pi^2 I^2_{\frac{1}{3}} + \sqrt{3}\,\pi K_{\frac{1}{3}} I_{\frac{1}{3}}}$$

$$= \frac{3\pi}{4\mu} \frac{1}{K^2_{\frac{1}{3}} + \pi^2 I^2_{\frac{1}{3}} + \sqrt{3}\,\pi K_{\frac{1}{3}} I_{\frac{1}{3}}}.$$

Also

$$\phi(x,\lambda) = -\frac{2\pi}{3\sqrt{3}} \lambda^{\frac{1}{2}} (x+\lambda)^{\frac{1}{2}} \{J_{\frac{1}{3}}(X) J_{-\frac{1}{3}}(Z) - J_{-\frac{1}{3}}(X) J_{\frac{1}{3}}(Z)\}.$$

Hence, if $-x < \lambda < 0$,

$$\phi(x,\lambda) = -\frac{2\pi}{3\sqrt{3}} \mu^{\frac{1}{2}} (x+\lambda)^{\frac{1}{2}} \{J_{\frac{1}{3}}(X) I_{-\frac{1}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}}) + J_{-\frac{1}{3}}(X) I_{\frac{1}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}})\},$$

and if $\lambda < -x$,

$$\phi(x,\lambda) = \frac{2\pi}{3\sqrt{3}}\mu^{\frac{1}{2}}(\mu-x)^{\frac{1}{2}}[I_{\frac{1}{3}}\{\tfrac{2}{3}(\mu-x)^{\frac{3}{2}}\}I_{-\frac{1}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}})-I_{-\frac{1}{3}}\{\tfrac{2}{3}(\mu-x)^{\frac{3}{2}}\}I_{\frac{1}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}})].$$

The expansion formula can then be written down from (3.1.1).

A similar result may be obtained with an interval $(-\infty,\infty)$. We obtain

$$\Phi(x,\lambda) = \frac{\pi}{6i}(x+\lambda)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(X)\int\limits_{-\infty}^{x}(y+\lambda)^{\frac{1}{2}}H_{\frac{1}{3}}^{(2)}(Y)f(y)\,dy +$$

$$+\frac{\pi}{6i}(x+\lambda)^{\frac{1}{2}}H_{\frac{1}{3}}^{(2)}(X)\int\limits_{x}^{\infty}(y+\lambda)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(Y)f(y)\,dy,$$

etc.

4.14. Let $q(x) = -e^{2x}$ $(-\infty < x < \infty)$. Solutions of

$$\frac{d^2y}{dx^2}+(\lambda+e^{2x})y = 0$$

are $J_\nu(e^x)$, $J_{-\nu}(e^x)$, where $\nu = i\sqrt{\lambda}$. If $\mathbf{I}(\lambda) > 0$, the former only is $L^2(-\infty,0)$, but both are $L^2(0,\infty)$. It is therefore the limit-point case as $x \to -\infty$, the limit-circle case as $x \to \infty$.

Since $$J_\nu(z)J'_{-\nu}(z)-J'_\nu(z)J_{-\nu}(z) = -\frac{2\sin\nu\pi}{\pi z},$$

the solutions $\theta(x)$, $\phi(x)$ satisfying $\theta(0) = 1$, $\theta'(0) = 0$, $\phi(0) = 0$, $\phi'(0) = -1$ are

$$\theta(x) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J'_{-\nu}(1)-J_{-\nu}(e^x)J'_\nu(1)\},$$

$$\phi(x) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J_{-\nu}(1)-J_{-\nu}(e^x)J_\nu(1)\}.$$

Hence $$m_1(\lambda) = -J'_\nu(1)/J_\nu(1),$$

$$\psi_1(x,\lambda) = \theta(x)+m_1(\lambda)\phi(x) = -J_\nu(e^x)/J_\nu(1).$$

Now consider the interval $(0,b)$. In the notation of § 2.1, we have

$$-l(\lambda) = \frac{\{J_\nu(e^b)J'_{-\nu}(1)-J_{-\nu}(e^b)J'_\nu(1)\}\cot\beta+e^b\{J'_\nu(e^b)J'_{-\nu}(1)-J'_{-\nu}(e^b)J'_\nu(1)\}}{\{J_\nu(e^b)J_{-\nu}(1)-J_{-\nu}(e^b)J_\nu(1)\}\cot\beta+e^b\{J'_\nu(e^b)J_{-\nu}(1)-J'_{-\nu}(e^b)J_\nu(1)\}}.$$

As $b \to \infty$, the denominator is asymptotic to

$$\left(\frac{2}{\pi e^b}\right)^{\frac{1}{2}}\{J_{-\nu}(1)\cos(e^b-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)-J_\nu(1)\cos(e^b+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\}\cot\beta-$$
$$-\left(\frac{2e^b}{\pi}\right)^{\frac{1}{2}}\{J_{-\nu}(1)\sin(e^b-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)-J_\nu(1)\sin(e^b+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\},$$

and similarly for the numerator. Hence

$$-l(\lambda) \sim$$
$$\frac{\{J'_{-\nu}(1)-J'_\nu(1)\}\cos\tfrac{1}{2}\nu\pi\cos(e^b+K)+\{J'_{-\nu}(1)+J'_\nu(1)\}\sin\tfrac{1}{2}\nu\pi\sin(e^b+K)}{\{J_{-\nu}(1)-J_\nu(1)\}\cos\tfrac{1}{2}\nu\pi\cos(e^b+K)+\{J_{-\nu}(1)+J_\nu(1)\}\sin\tfrac{1}{2}\nu\pi\sin(e^b+K)},$$

where $\cot K = e^{-b}\cot\beta$. The limit-circle is obtained by giving $\cot(e^b+K)$ any constant value C in this formula. Take, for example, $C = 0$. Then

$$m_2(\lambda) = -\frac{J'_\nu(1)+J'_{-\nu}(1)}{J_\nu(1)+J_{-\nu}(1)},$$

$$\psi_2(x,\lambda) = -\frac{J_\nu(e^x)+J_{-\nu}(e^x)}{J_\nu(1)+J_{-\nu}(1)}.$$

Hence

$$\Phi(x,\lambda) = \frac{\pi}{2\sin(i\pi\sqrt{\lambda})}\left[\{J_{i\sqrt{\lambda}}(e^x)+J_{-i\sqrt{\lambda}}(e^x)\}\int_{-\infty}^{x} J_{i\sqrt{\lambda}}(e^y)f(y)\,dy+\right.$$
$$\left.+J_{i\sqrt{\lambda}}(e^x)\int_{x}^{\infty}\{J_{i\sqrt{\lambda}}(e^y)+J_{-i\sqrt{\lambda}}(e^y)\}f(y)\,dy\right]$$

and the expansion formula is

$$f(x) = \int_0^\infty \frac{J_{i\sqrt{\lambda}}(e^x)+J_{-i\sqrt{\lambda}}(e^x)}{4\sinh(\pi\sqrt{\lambda})}\,d\lambda\int_{-\infty}^{\infty}\{J_{i\sqrt{\lambda}}(e^y)+J_{-i\sqrt{\lambda}}(e^y)\}f(y)\,dy.$$

4.15. The Laguerre or Sonine polynomials. The Laguerre polynomial of order n is defined by

$$L_n(X) = \frac{e^X}{n!}\left(\frac{d}{dX}\right)^n(e^{-X}X^n). \tag{4.15.1}$$

The generalized Laguerre polynomials are

$$L_n^{(\alpha)}(X) = \frac{e^X X^{-\alpha}}{n!}\left(\frac{d}{dX}\right)^n(e^{-X}X^{n+\alpha})$$
$$= \sum_{r=0}^{n}\frac{\Gamma(n+\alpha+1)}{r!\,(n-r)!\,\Gamma(n-r+\alpha+1)}(-X)^{n-r}. \tag{4.15.2}$$

Thus $$L_n^{(\alpha)}(X) = (-1)^n\Gamma(n+\alpha+1)T_\alpha^n(X),$$

where the $T_\alpha^n(X)$ are the Sonine polynomials.

Now if $u = e^{-X}X^{n+\alpha}$, then

$$X\frac{du}{dX} = (n+\alpha-X)u.$$

Differentiating $n+1$ times,

$$X\frac{d^{n+2}u}{dX^{n+2}}+(n+1)\frac{d^{n+1}u}{dX^{n+1}} = (n+\alpha-X)\frac{d^{n+1}u}{dX^{n+1}}-(n+1)\frac{d^{n}u}{dX^{n}}.$$

Hence $u_1 = (d/dX)^n(e^{-X}X^{n+\alpha})$ satisfies the differential equation

$$X\frac{d^2u_1}{dX^2}+(X+1-\alpha)\frac{du_1}{dX}+(n+1)u_1 = 0,$$

and $u_2 = e^X u_1$ satisfies

$$\frac{d^2u_2}{dX^2}-\left(1+\frac{\alpha-1}{X}\right)\frac{du_2}{dX}+\frac{n+\alpha}{X}u_2 = 0. \tag{4.15.3}$$

This is therefore the equation satisfied by $X^\alpha L_n^{(\alpha)}(X)$.

Putting $X = x^2$, $u_3 = x^{-\alpha+\frac{1}{2}}e^{-\frac{1}{2}x^2}u_2$, we obtain

$$\frac{d^2u_3}{dx^2} = \left(x^2+\frac{\alpha^2-\frac{1}{4}}{x^2}-4n-2\alpha-2\right)u_3. \tag{4.15.4}$$

We should therefore expect to obtain the functions

$$x^{\alpha+\frac{1}{2}}e^{-\frac{1}{2}x^2}L_n^{(\alpha)}(x^2)$$

as the eigenfunctions associated with

$$q(x) = x^2+\frac{\alpha^2-\frac{1}{4}}{x^2} \quad (0 < x < \infty). \tag{4.15.5}$$

Consider then the equation

$$\frac{d^2y}{dx^2}+\left(\lambda-x^2-\frac{\alpha^2-\frac{1}{4}}{x^2}\right)y = 0. \tag{4.15.6}$$

This is equivalent to (4.15.4) if $\lambda = 4n+2\alpha+2$. Hence $Y = x^{\alpha-\frac{1}{2}}e^{\frac{1}{2}x^2}y$ satisfies

$$\frac{d^2Y}{dX^2}-\left(1+\frac{\alpha-1}{X}\right)\frac{dY}{dX}+\frac{\lambda+2\alpha-2}{4X}Y = 0 \tag{4.15.7}$$

corresponding to (4.15.3).

Assume as a solution

$$Y = \int g(z)e^{Xz}\,dz$$

taken round a suitable contour. Integrating by parts,

$$Y = -\frac{1}{X}\int g'(z)e^{Xz}\,dz,$$

if the integrated term vanishes at the limits. Similarly

$$\frac{dY}{dX} = \int zg(z)e^{Xz}\,dz = -\frac{1}{X}\int \{zg'(z)+g(z)\}e^{Xz}\,dz,$$

$$\frac{d^2Y}{dX^2} = \int z^2g(z)e^{Xz}\,dz = -\frac{1}{X}\int \{z^2g'(z)+2zg(z)\}e^{Xz}\,dz.$$

Hence (4.15.7) gives

$$-\frac{1}{X}\int \{z^2g'(z)+2zg(z)-zg'(z)-g(z)+(\alpha-1)zg(z)-$$
$$-\tfrac{1}{4}(\lambda+2\alpha-2)g(z)\}e^{Xz}\,dz = 0.$$

This is true if

$$\frac{g'(z)}{g(z)} = \frac{-(\alpha+1)z+\frac{1}{4}(\lambda+2\alpha+2)}{z(z-1)} = \frac{\lambda-2\alpha-2}{4(z-1)} - \frac{\lambda+2\alpha+2}{4z},$$

$$g(z) = (z-1)^{\frac{1}{4}(\lambda-2\alpha-2)}z^{-\frac{1}{4}(\lambda+2\alpha+2)}.$$

Since

$$e^{-\frac{1}{2}X}Y = \int g(z)e^{X(z-\frac{1}{2})}\,dz = \int g(z'+\tfrac{1}{2})e^{Xz'}\,dz'$$

we obtain finally as solutions of (4.15.6)

$$\phi_1(x,\lambda) = x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(\frac{1}{2}+)} (z-\tfrac{1}{2})^{\frac{1}{4}(\lambda-2\alpha-2)}(z+\tfrac{1}{2})^{-\frac{1}{4}(\lambda+2\alpha+2)}e^{x^2z}\,dz \tag{4.15.8}$$

and

$$\phi_2(x,\lambda) = x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(-\frac{1}{2}+)} (z-\tfrac{1}{2})^{\frac{1}{4}(\lambda-2\alpha-2)}(z+\tfrac{1}{2})^{-\frac{1}{4}(\lambda+2\alpha+2)}e^{x^2z}\,dz. \tag{4.15.9}$$

Here the integrands have the value which is real and positive for z real and greater than $\frac{1}{2}$, and λ real.

As $x \to \infty$, ϕ_1 is dominated by the part of the integral near $z = \frac{1}{2}$, and so is asymptotic to

$$x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(\frac{1}{2}+)} (z-\tfrac{1}{2})^{\frac{1}{4}(\lambda-2\alpha-2)}e^{x^2z}\,dz = x^{\frac{1}{2}-\alpha}e^{\frac{1}{2}x^2}\int_{-\infty}^{(0+)} z^{\frac{1}{4}(\lambda-2\alpha-2)}e^{x^2z}\,dz$$
$$= x^{-\frac{1}{2}\lambda-\frac{1}{2}}e^{\frac{1}{2}x^2}2i\sin\{\tfrac{1}{4}\pi(\lambda-2\alpha-2)\}\Gamma\{\tfrac{1}{4}(\lambda-2\alpha+2)\}.$$

Similarly

$$\phi_2 \sim x^{\frac{1}{2}-\alpha}e^{\frac{1}{4}i\pi(\lambda-2\alpha-2)}\int_{-\infty}^{(-\frac{1}{2}+)} (z+\tfrac{1}{2})^{-\frac{1}{4}(\lambda+2\alpha+2)}e^{x^2z}\,dz$$
$$= x^{\frac{1}{2}\lambda-\frac{1}{2}}e^{-\frac{1}{2}x^2}2ie^{\frac{1}{4}i\pi(\lambda-2\alpha-2)}\sin\{\tfrac{1}{4}\pi(-\lambda-2\alpha-2)\}\Gamma\{\tfrac{1}{4}(-\lambda-2\alpha+2)\}.$$

Also
$$\phi_1' \sim x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(\frac{1}{2}+)} (z-\tfrac{1}{2})^{\frac{1}{4}(\lambda-2\alpha-2)}2xze^{x^2z}\,dz \sim x\phi_1$$
and similarly
$$\phi_2' \sim -x\phi_2.$$
Hence
$$W(\phi_1,\phi_2) \sim -2x\phi_1\phi_2$$
$$\sim 8e^{\frac{1}{4}i\pi(\lambda-2\alpha-2)}\sin\{\tfrac{1}{4}\pi(\lambda-2\alpha-2)\}\sin\{\tfrac{1}{4}\pi(-\lambda-2\alpha-2)\}\times$$
$$\times\Gamma\{\tfrac{1}{4}(\lambda-2\alpha+2)\}\Gamma\{\tfrac{1}{4}(-\lambda+2\alpha-2)\},$$
i.e.
$$W(\phi_1,\phi_2) = \frac{8\pi^2e^{\frac{1}{4}i\pi(\lambda-2\alpha-2)}}{\Gamma\{\frac{1}{4}(2\alpha+2-\lambda)\}\Gamma\{\frac{1}{4}(2\alpha+2+\lambda)\}}.$$
We have
$$\Phi(x,\lambda) = \frac{1}{W(\phi_1,\phi_2)}\Big\{\phi_2(x,\lambda)\int_0^x \phi_1(y,\lambda)f(y)\,dy+$$
$$+\phi_1(x,\lambda)\int_x^\infty \phi_2(y,\lambda)f(y)\,dy\Big\}.$$

Now $1/W(\phi_1,\phi_2)$ has poles at the points
$$\lambda_n = \pm(4n+2\alpha+2) \quad (n = 0, 1,\ldots).$$
Taking the upper sign
$$\phi_1(x,\lambda_n) = x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(\frac{1}{2}+)} \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+\alpha+1}}e^{x^2z}\,dz.$$
In this case there is no singularity at $z = \frac{1}{2}$, so that
$$\phi_1(x,\lambda_n) = x^{\frac{1}{2}-\alpha}\int_{-\infty}^{(-\frac{1}{2}+)} \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+\alpha+1}}e^{x^2z}\,dz = x^{\frac{1}{2}-\alpha}e^{-\frac{1}{2}x^2}\int_{-\infty}^{(0+)} \frac{(z-1)^n}{z^{n+\alpha+1}}e^{x^2z}\,dz$$
$$= x^{\frac{1}{2}-\alpha}e^{-\frac{1}{2}x^2}\sum_{r=0}^{n}\frac{(-1)^{n-r}n!}{r!(n-r)!}\int_{-\infty}^{(0+)} z^{r-n-\alpha-1}e^{x^2z}\,dz$$
$$= x^{\frac{1}{2}-\alpha}e^{-\frac{1}{2}x^2}\sum_{r=0}^{n}\frac{(-1)^{n-r}n!}{r!\,(n-r)!}(-1)^{n-r}2i\sin\pi\alpha\times$$
$$\times\Gamma(r-n-\alpha)x^{2n-2r+2\alpha}$$
$$= -x^{\alpha+\frac{1}{2}}e^{-\frac{1}{2}x^2}2i\pi\sum_{r=0}^{n}\frac{(-1)^{n-r}x^{2n-2r}n!}{r!\,(n-r)!\,\Gamma(n-r+\alpha+1)}$$
$$= \frac{-2i\pi\,n!}{\Gamma(n+\alpha+1)}x^{\alpha+\frac{1}{2}}e^{-\frac{1}{2}x^2}L_n^{(\alpha)}(x^2),$$

and $$\phi_2(x,\lambda_n) = \phi_1(x,\lambda_n).$$

Taking the lower sign

$$\phi_2(x,\lambda_n) = x^{\frac{1}{2}-\alpha} \int\limits_{-\infty}^{(-\frac{1}{2}+)} \frac{(z-\frac{1}{2})^{-n-\alpha-1}}{(z+\frac{1}{2})^{-n}} e^{x^2 z}\, dz = 0,$$

so that $\Phi(x,\lambda)$ is regular at these points.

The residue of $1/W(\phi_1,\phi_2)$ at $\lambda = 4n+2\alpha+2$ is $-\Gamma(n+\alpha+1)/2\pi^2 n!$ Hence the residue of $\Phi(x,\lambda)$ is

$$\frac{\Gamma(n+\alpha+1)}{n!\,\pi^2} \frac{2\pi^2(n!)^2}{\Gamma^2(n+\alpha+1)} x^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2) \int\limits_0^\infty y^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{(\alpha)}(y^2) f(y)\, dy$$

$$= \frac{2.n!}{\Gamma(n+\alpha+1)} x^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2) \int\limits_0^\infty y^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{(\alpha)}(y^2) f(y)\, dy.$$

4.16. The 'hydrogen atom'. Let

$$q(x) = \frac{r(r+1)}{x^2} - \frac{c}{x} \quad (0 < x < \infty), \tag{4.16.1}$$

where r is zero or a positive integer. Then

$$\frac{d^2y}{dx^2} + \left\{\lambda + \frac{c}{x} - \frac{r(r+1)}{x^2}\right\} y = 0. \tag{4.16.2}$$

This is the celebrated equation from which physicists have been able to derive the theory of the hydrogen atom.

Putting $y = x^{-r}Y$, $\lambda = s^2$, we obtain

$$\frac{d^2Y}{dx^2} - \frac{2r}{x}\frac{dY}{dx} + \left(\frac{c}{x} + s^2\right) Y = 0. \tag{4.16.3}$$

Assuming $$Y = \int g(z) e^{xz}\, dz$$

and proceeding as in the last section, we find that Y satisfies (4.16.3) if

$$\frac{g'(z)}{g(z)} = \frac{c-2(r+1)z}{z^2+s^2} = \frac{k-r-1}{z+is} - \frac{k+r+1}{z-is},$$

where $k = \frac{1}{2}ic/s$. Hence

$$g(z) = (z+is)^{k-r-1}(z-is)^{-k-r-1}.$$

Hence solutions of (4.16.2) are

$$y = \frac{1}{x^r}\int (z+is)^{k-r-1}(z-is)^{-k-r-1}e^{xz}\,dz$$

taken round suitable contours.

Let $$\phi_1(x,\lambda) = \frac{1}{x^r}\int_C (z+is)^{k-r-1}(z-is)^{-k-r-1}e^{xz}\,dz, \tag{4.16.4}$$

where C is a closed contour surrounding both is and $-is$, and where $\arg(z+is)$ and $\arg(z-is)$ both go from $-\pi$ to π round the contour. Thus $(z+is)^k$ is multiplied by $e^{2\pi ik}$, and $(z-is)^{-k}$ by $e^{-2\pi ik}$, and the whole integrand returns to its original value.

Let $$\phi_2(x,\lambda) = \frac{1}{x^r}\int_{-\infty}^{(is+)} (z+is)^{k-r-1}(z-is)^{-k-r-1}e^{xz}\,dz, \tag{4.16.5}$$

where the loop includes is but not $-is$. Here $\arg(z-is)$ goes from $-\pi$ to π, and, for s in the first quadrant, $\arg(z+is)$ starts with the value π and returns to π again.

Putting $z = is+\zeta/x$,

$$\phi_2(x,\lambda) = x^k(2is)^{k-r-1}e^{ixs}\int_{-\infty}^{(0+)} \left(1+\frac{\zeta}{2isx}\right)^{k-r-1}\zeta^{-k-r-1}e^{\zeta}\,d\zeta,$$

where $\arg(1+\zeta/2isx) \to 0$ as $\zeta \to 0$. Hence by Whittaker and Watson, *Modern Analysis*, § 16.12,

$$\phi_2(x,\lambda) = \frac{2\pi ie^{i\pi k}}{\Gamma(k+r+1)}(2is)^{-r-1}W_{k,-r-\frac{1}{2}}(-2isx).$$

The integral in ϕ_1 clearly represents a function regular near $x = 0$. The coefficients of all powers of x up to x^{2r} vanish, as is seen on expanding the contour to infinity. The coefficient of x^{2r+1} is

$$\frac{1}{(2r+1)!}\int_C (z+is)^{k-r-1}(z-is)^{-k-r-1}z^{2r+1}\,dz = \frac{2\pi i}{(2r+1)!}$$

(again expanding the contour to infinity). Since $M_{k,r+\frac{1}{2}}(z) \sim z^{r+1}$ for small z, it follows that

$$\phi_1(x,\lambda) = \frac{2\pi i}{(2r+1)!}\,\frac{M_{k,r+\frac{1}{2}}(-2isx)}{(-2is)^{r+1}}.$$

Let $\omega(\lambda) = W(\phi_1,\phi_2)$. Denoting the integrals in (4.16.4) and (4.16.5) by I_1 and I_2,

$$\omega(\lambda) = x^{-2r}W(I_1,I_2).$$

Hence $$(2r)!\,\omega(\lambda) = \left[\left(\frac{d}{dx}\right)^{2r} W(I_1, I_2)\right]_{x=0}.$$

As above we have $I_1^{(\nu)}(0) = 0$ for $\nu = 1,\ldots, 2r$, and $I_1^{(2r+1)}(0) = 2\pi i$. Hence

$$(2r)!\,\omega(\lambda) = -2\pi i I_2(0)$$

$$= -2\pi i \int\limits_{-\infty}^{(is+)} (z+is)^{k-r-1}(z-is)^{-k-r-1}\,dz$$

$$= -2\pi i \int\limits_{-\infty}^{(0+)} (\zeta+2is)^{k-r-1}\zeta^{-k-r-1}\,d\zeta.$$

Let $\arg(is) = \theta$ $(\frac{1}{2}\pi < \theta < \pi)$. We may turn the contour into a position in which ζ comes from infinity with $\arg\zeta = \theta-2\pi$, passes round the origin, and returns to infinity with $\arg\zeta = \theta$; and $\arg(\zeta+2is) = \theta$. Hence the last integral is equal to

$$e^{i\theta(k-r-1)}(e^{i\theta(-k-r)}-e^{i(\theta-2\pi)(-k-r)})|2is|^{-2r-1}\int\limits_0^\infty (\xi+1)^{k-r-1}\xi^{-k-r-1}\,d\xi$$

$$= (2is)^{-2r-1}(1-e^{2i\pi k})\frac{\Gamma(-r-k)\Gamma(2r+1)}{\Gamma(r+1-k)}.$$

Hence $$\omega(\lambda) = \frac{-2\pi i(2is)^{-2r-1}(1-e^{2i\pi k})}{(r-k)(r-k-1)\ldots(-r-k)}.$$

Now

$$\Phi(x,\lambda) = \frac{\phi_2(x)}{\omega(\lambda)}\int\limits_0^x \phi_1(y)f(y)\,dy + \frac{\phi_1(x)}{\omega(\lambda)}\int\limits_x^\infty \phi_2(y)f(y)\,dy.$$

This has poles at the zeros of $\omega(\lambda)$, i.e. at

$$k = r+n+1, \qquad \lambda = \lambda_n = -\frac{c^2}{4(r+n+1)^2} \quad (n = 0, 1,\ldots).$$

Now

$$\omega'(\lambda_n) = \frac{-4\pi^2(2is_n)^{-2r-1}}{(-n-1)\ldots(-n-1-2r)}\frac{dk}{d\lambda}$$

$$= \frac{4\pi^2 n!}{(n+2r+1)!}\left(-\frac{n+r+1}{c}\right)^{2r+1}\frac{2(n+r+1)^3}{c^2}$$

$$= -\frac{8\pi^2 n!}{(n+2r+1)!}\frac{(n+r+1)^{2r+4}}{c^{2r+3}}.$$

Also

$$\phi_1(x,\lambda_n) = \phi_2(x,\lambda_n) = \frac{1}{x^r}\int (z+is_n)^n(z-is_n)^{-n-2r-2}e^{xz}\,dz$$

$$= x^{r+1}e^{-cx/2(n+r+1)}\int (\zeta+2ixs_n)^n\zeta^{-n-2r-2}e^{\zeta}\,d\zeta,$$

where $z = is_n + \zeta/x$, and the contour is now a closed curve round the pole. From (4.15.1) and Cauchy's formula for the pth derivative

$$L_p(t) = e^t \frac{p!}{2\pi i} \int \frac{z^p e^{-z}}{(z-t)^{p+1}}\, dz = \frac{p!}{2\pi i} \int \frac{(\zeta+t)^p e^{-\zeta}}{\zeta^{p+1}}\, d\zeta.$$

Hence
$$L_p^m(t) = \frac{(p!)^2}{2\pi i(p-m)!} \int \frac{(\zeta+t)^{p-m} e^{-\zeta}}{\zeta^{p+1}}\, d\zeta,$$

where $L_p^m(t)$ denotes $(d/dt)^m L_p(t)$. Hence

$$\phi_1(x, \lambda_n) = x^{r+1} e^{-cx/2(n+r+1)} \frac{2\pi i n!}{\{(n+2r+1)!\}^2} L_{n+2r+1}^{2r+1}\left(\frac{cx}{n+r+1}\right).$$

Hence $\Phi(x, \lambda)$ has the residue

$$\frac{c^{2r+3}}{2(n+r+1)^{2r+4}} \frac{n!}{\{(n+2r+1)!\}^3} x^{r+1} e^{-cx/(n+r+1)} L_{n+2r+1}^{2r+1}\left(\frac{cx}{n+r+1}\right) \times$$
$$\times \int_0^\infty y^{r+1} e^{-cy/(n+r+1)} L_{n+2r+1}^{2r+1}\left(\frac{cy}{n+r+1}\right) f(y)\, dy.$$

The sum of these terms is the contribution of the part of the spectrum with $\lambda < 0$.

We can write

$$\phi_1(x, \lambda) = \frac{1}{x^r} \int_{-\infty}^{(-is+)} (z+is)^{k-r-1}(z-is)^{-k-r-1} e^{xz}\, dz +$$
$$+ \frac{1}{x^r} \int_{-\infty}^{(is+)} (z+is)^{k-r-1}(z-is)^{-k-r-1} e^{xz}\, dz,$$

where in the first integral $\arg(z+is)$ goes from $-\pi$ to π, $\arg(z-is)$ from $-\pi$ to $-\pi$; in the second, $\arg(z+is)$ goes from π to π, $\arg(z-is)$ from $-\pi$ to π. The second term is thus $\phi_2(x, \lambda)$. Now if in $\phi_2(x,\lambda)$ we increase $\arg s$ by π, we get an expression apparently the same as the above first term, but with $\arg(z-is)$ going from π to π instead of from $-\pi$ to $-\pi$. Hence

$$\phi_1(x, \lambda) = e^{+2\pi i k} \phi_2(x, \lambda e^{2i\pi}) + \phi_2(x, \lambda).$$

Also
$$\omega(\lambda e^{2i\pi}) = -e^{-2\pi i k} \omega(\lambda).$$

Hence for λ real and positive

$$\frac{\phi_2(x, \lambda)}{\omega(\lambda)} - \frac{\phi_2(x, \lambda e^{2\pi i})}{\omega(\lambda e^{2\pi i})} = \frac{\phi_2(x, \lambda) + e^{2\pi i k} \phi_2(x, \lambda e^{2\pi i})}{\omega(\lambda)} = \frac{\phi_1(x, \lambda)}{\omega(\lambda)}.$$

Hence

$$\mathbf{I}\Phi(x,\lambda) = \frac{1}{2i}\{\Phi(x,\lambda)-\Phi(x,\lambda e^{2\pi i})\} = \frac{1}{2i}\frac{\phi_1(x,\lambda)}{\omega(\lambda)}\int\limits_0^\infty \phi_1(y,\lambda)f(y)\,dy,$$

where

$$i\omega(\lambda) = \frac{2\pi(2s)^{-2r-1}(1-e^{-\pi c/s})}{(r^2+c^2/4s^2)...(1+c^2/4s^2)(c/2s)}$$
$$= \frac{2\pi(1-e^{-\pi c/s})}{(4s^2+c^2)...(4r^2s^2+c^2)c}.$$

Hence the interval $0<\lambda<\infty$ contributes a term

$$\frac{1}{4\pi^2}\int\limits_0^\infty \frac{(4s^2+c^2)...(4r^2s^2+c^2)c}{1-e^{-\pi c/s}}\phi_1(x,s^2)2s\,ds\int\limits_0^\infty \phi_1(y,s^2)f(y)\,dy.$$

4.17. Hypergeometric functions. Consider the equation

$$X(X+1)\frac{d^2Y}{dX^2}+\{\gamma+(\alpha+1)X\}\frac{dY}{dX}+\lambda^*Y = 0 \quad (0<X<\infty). \quad (4.17.1)$$

Solutions are

$$Y_1 = F(a,b,c,-X), \qquad Y_2 = X^{1-c}F(a-c+1,b-c+1,2-c,-X),$$

where $\quad c=\gamma, \qquad a+b=\alpha, \qquad ab=\lambda^*,$

say $\quad a = \tfrac{1}{2}\alpha+\tfrac{1}{2}(\alpha^2-4\lambda^*)^{\frac{1}{2}}, \qquad b = \tfrac{1}{2}\alpha-\tfrac{1}{2}(\alpha^2-4\lambda^*)^{\frac{1}{2}}.$

We transform into the standard form as in § 4.3. Putting

$$X = \sinh^2\tfrac{1}{2}x,$$

we obtain

$$\frac{d^2Y}{dx^2}+\beta(x)\frac{dY}{dx}+\lambda^*Y = 0, \quad (4.17.2)$$

where

$$\beta(x) = \frac{\gamma+(\alpha+1)X-(X+\frac{1}{2})}{\{X(X+1)\}^{\frac{1}{2}}} = \frac{\alpha\cosh x+2\gamma-1-\alpha}{\sinh x}.$$

Putting $Y=r(x)y$, where

$$r(x) = \left(\frac{e^x-1}{e^x+1}\right)^{\frac{1}{2}+\frac{1}{2}\alpha-\gamma}\sinh^{-\frac{1}{2}\alpha}x,$$

we obtain

$$\frac{d^2y}{dx^2}+\{\lambda^*-q^*(x)\}y = 0, \quad (4.17.3)$$

where

$$q^*(x) = \tfrac{1}{4}\beta^2(x)+\tfrac{1}{2}\beta'(x)$$
$$= \frac{\alpha^2\cosh^2x+2(\alpha-1)(2\gamma-1-\alpha)\cosh x+(2\gamma-1-\alpha)^2-2\alpha}{4\sinh^2x}.$$

Finally, put $\lambda = \lambda^* - \frac{1}{4}\alpha^2$, and

$$q(x) = q^*(x) - \tfrac{1}{4}\alpha^2$$

$$= \frac{2(\alpha-1)(2\gamma-1-\alpha)\cosh x + 2\alpha^2 - 4\gamma\alpha + (1-2\gamma)^2}{4\sinh^2 x}; \tag{4.17.4}$$

we obtain

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0. \tag{4.17.5}$$

In this notation $a = \frac{1}{2}\alpha + i\sqrt{\lambda}$, $b = \frac{1}{2}\alpha - i\sqrt{\lambda}$. Corresponding to Y_1 and Y_2 we obtain solutions of (4.17.5) asymptotic to

$$Ax^{\gamma-\frac{1}{2}}, \qquad A(x^2)^{1-\gamma}x^{\gamma-\frac{1}{2}} = Ax^{\frac{3}{2}-\gamma}$$

as $x \to 0$. Hence the origin is of limit-circle type if $0 < \gamma < 2$, and otherwise of limit-point type. To take a definite case, let $\gamma > 2$. Then the only solution of (4.17.5) belonging to $L^2(0,1)$ is

$$y = \eta_1(x,\lambda) = \{r(x)\}^{-1}F(\tfrac{1}{2}\alpha + i\sqrt{\lambda}, \tfrac{1}{2}\alpha - i\sqrt{\lambda}, \gamma, -\sinh^2\tfrac{1}{2}x).$$

A solution of (4.17.1) for $X > 1$ is

$$X^{-b}F(b, 1-c+b, 1-a+b, -1/X).$$

This leads to

$$y = \eta_2(x,\lambda) = \{r(x)\}^{-1}\sinh^{-2b}\tfrac{1}{2}x\,F(b, 1-c+b, 1-a+b, -\sinh^{-2}\tfrac{1}{2}x)$$

as a solution of (4.17.5).

As $x \to \infty$,

$$\eta_2(x,\lambda) \sim \sinh^{\frac{1}{2}\alpha}x\,\sinh^{-2b}\tfrac{1}{2}x \sim Ae^{ix\sqrt{\lambda}}.$$

Hence η_2 is $L^2(1,\infty)$ if $0 < \arg\sqrt{\lambda} < \frac{1}{2}\pi$.

The other solution, obtained by interchanging a and b, gives $e^{-ix\sqrt{\lambda}}$, and so is not L^2. Also

$$\begin{aligned}
&W(\eta_1, \eta_2)\\
&= \{r(x)\}^{-2}W\{F(a,b,c,-\sinh^2\tfrac{1}{2}x),\\
&\qquad\qquad \sinh^{-2b}\tfrac{1}{2}x\,F(b,1-c+b,1-a+b,-\sinh^{-2}\tfrac{1}{2}x)\}\\
&= \{r(x)\}^{-2}\{X(X+1)\}^{\frac{1}{2}}W\{F(a,b,c,-X),\\
&\qquad\qquad X^{-b}F(b,1-c+b,1-a+b,-1/X)\}\\
&= \frac{\{X(X+1)\}^{\frac{1}{2}}}{\{r(x)\}^2}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}W\{X^{-a}F(a,1-c+a,1-b+a,-1/X),\\
&\qquad\qquad X^{-b}F(b,1-c+b,1-a+b,-1/X)\}\\
&\sim 2^{a+b}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(a-b) = \frac{2^{\alpha+1}i\sqrt{\lambda}\,\Gamma(\gamma)\Gamma(-2i\sqrt{\lambda})}{\Gamma(\frac{1}{2}\alpha - i\sqrt{\lambda})\Gamma(\gamma-\frac{1}{2}\alpha-i\sqrt{\lambda})}
\end{aligned}$$

as $X \to \infty$. Since $W(\eta_1, \eta_2)$ is independent of x, this is its value. Hence

$$\Phi(x,\lambda) = \frac{\Gamma(\frac{1}{2}\alpha - i\sqrt{\lambda})\Gamma(\gamma - \frac{1}{2}\alpha - i\sqrt{\lambda})}{2^{\alpha+1} i\sqrt{\lambda}\,\Gamma(\gamma)\Gamma(-2i\sqrt{\lambda})} \times$$

$$\times \left\{\eta_2(x,\lambda)\int_0^x \eta_1(y,\lambda)f(y)\,dy + \eta_1(x,\lambda)\int_x^\infty \eta_2(y,\lambda)f(y)\,dy\right\}.$$

When λ is real and positive, η_1 is real, and since a and b are conjugates

$$\mathbf{I}\left\{\frac{\eta_2(x,\lambda)}{W(\eta_1,\eta_2)}\right\} = \frac{1}{2^{\alpha+1} i r(x)} \times$$

$$\times \left\{\frac{\Gamma(b)\Gamma(c-a)}{\Gamma(c)\Gamma(b-a)(a-b)} X^{-b} F(b, 1-c+b, 1-a+b, -1/X) - \right.$$

$$\left. - \frac{\Gamma(a)\Gamma(c-b)}{\Gamma(c)\Gamma(a-b)(b-a)} X^{-a} F(a, 1-c+a, 1-b+a, -1/X)\right\}$$

$$= \frac{1}{2^{\alpha+1} r(x) i(a-b)} \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma^2(c)\Gamma(b-a)\Gamma(a-b)} F(a,b,c,-X)$$

$$= -\frac{1}{2^{\alpha+2}\sqrt{\lambda}} \left|\frac{\Gamma(\frac{1}{2}\alpha + i\sqrt{\lambda})\Gamma(\gamma - \frac{1}{2}\alpha + i\sqrt{\lambda})}{\Gamma(\gamma)\Gamma(2i\sqrt{\lambda})}\right|^2 \eta_1(x,\lambda).$$

Hence there is a continuous spectrum from 0 to ∞, which contributes to the expansion of $f(x)$

$$\frac{1}{2^{\alpha+2}\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}\alpha + i\sqrt{\lambda})\Gamma(\gamma - \frac{1}{2}\alpha + i\sqrt{\lambda})}{\Gamma(\gamma)\Gamma(2i\sqrt{\lambda})}\right|^2 \eta_1(x,\lambda)\frac{d\lambda}{\sqrt{\lambda}} \int_0^\infty \eta_1(y,\lambda)f(y)\,dy.$$

There may also be a finite number of poles on the negative real axis. These will be at the points $-i\sqrt{\lambda} = -\frac{1}{2}\alpha - n$ or $-i\sqrt{\lambda} = \frac{1}{2}\alpha - \gamma - n$, if these are positive for any positive integer values of n. The corresponding residues are easily calculated.

4.18. The case $\gamma = \frac{1}{2}$. In this case

$$q(x) = \frac{\alpha(1-\alpha)}{4\cosh^2 \frac{1}{2}x}, \tag{4.18.1}$$

and there is no singularity at $x = 0$. The x-interval may be taken as $(-\infty, \infty)$. We have

$$r(x) = 2^{-\frac{1}{2}\alpha} \cosh^{-\alpha} \tfrac{1}{2}x,$$

and solutions of (4.17.5) are

$$\theta(x) = \cosh^\alpha \tfrac{1}{2}x\, F(\tfrac{1}{2}\alpha + i\sqrt{\lambda}, \tfrac{1}{2}\alpha - i\sqrt{\lambda}, \tfrac{1}{2}, -\sinh^2 \tfrac{1}{2}x),$$

$$\phi(x) = -2\cosh^\alpha \tfrac{1}{2}x \sinh \tfrac{1}{2}x\, F(\tfrac{1}{2} + \tfrac{1}{2}\alpha + i\sqrt{\lambda}, \tfrac{1}{2} + \tfrac{1}{2}\alpha - i\sqrt{\lambda}, \tfrac{3}{2}, -\sinh^2 \tfrac{1}{2}x).$$

These are even and odd respectively. The solution which is small as $x \to \infty$ is

$$\frac{\sinh^{-2b}\frac{1}{2}x}{2^{\frac{1}{2}\alpha}r(x)} F\left(b, b+\tfrac{1}{2}, 1-a+b, -\frac{1}{\sinh^2\frac{1}{2}x}\right)$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(1-2i\sqrt{\lambda})}{\Gamma(1-\frac{1}{2}\alpha-i\sqrt{\lambda})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha-i\sqrt{\lambda})}\theta(x)-$$
$$-\frac{\frac{1}{2}\Gamma(-\frac{1}{2})\Gamma(1-2i\sqrt{\lambda})}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha-i\sqrt{\lambda})\Gamma(\frac{1}{2}\alpha-i\sqrt{\lambda})}\phi(x).$$

Hence, in the notation of § 2.18 or § 3.1

$$m_2(\lambda) = \frac{\Gamma(1-\frac{1}{2}\alpha-i\sqrt{\lambda})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha-i\sqrt{\lambda})}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha-i\sqrt{\lambda})\Gamma(\frac{1}{2}\alpha-i\sqrt{\lambda})},$$

nd, since $q(x)$ is even, $m_1(\lambda) = -m_2(\lambda)$. In the notation of § 3.1

$$\xi'(\lambda) = \tfrac{1}{2}\mathbf{I}\{1/m_2(\lambda)\}, \qquad \zeta'(\lambda) = -\tfrac{1}{2}\mathbf{I}\{m_2(\lambda)\}.$$

There is a continuous spectrum from 0 to ∞, which contributes

$$\frac{1}{\pi}\int_0^\infty \theta(x,\lambda)\xi'(\lambda)\,d\lambda \int_{-\infty}^\infty \theta(y,\lambda)f(y)\,dy +$$
$$+\frac{1}{\pi}\int_0^\infty \phi(x,\lambda)\zeta'(\lambda)\,d\lambda \int_{-\infty}^\infty \phi(y,\lambda)f(y)\,dy$$

to the expansion of $f(x)$.

There may also be a finite number of poles on the negative real axis. Suppose, for example, that $\alpha = n+1$, where n is a positive integer. Then $m_2(\lambda)$ has zeros at the points

$$-i\sqrt{\lambda} = \tfrac{1}{2}n-r \quad (r = 0, 1,..., [\tfrac{1}{2}n]),$$

the residue of $1/m_2(\lambda)$ being

$$A_r = \frac{(-1)^r}{r!}\frac{(2r-n)\Gamma(n-r+\frac{1}{2})}{\Gamma(\frac{1}{2}-r)\Gamma(n-r+1)}.$$

This is case (ii) of § 2.18, with $\mu_2 = 1/A_r$, $\mu_1 = -1/A_r$. Hence $\Phi(x,\lambda)$ has the residue

$$-\tfrac{1}{2}A_r\,\theta(x,\lambda_r)\int_{-\infty}^\infty \theta(y,\lambda_r)f(y)\,dy,$$

where

$$\theta(x,\lambda_r) = \cosh^{n+1}\tfrac{1}{2}x\,F(r+\tfrac{1}{2}, n-r+\tfrac{1}{2}, \tfrac{1}{2}, -\sinh^2\tfrac{1}{2}x).$$

Actually the solutions of (4.17.5), where $q(x)$ is (4.18.1), are associated Legendre functions with argument $\tanh\frac{1}{2}x$. In fact

$$\theta(x,\lambda_r) = (-1)^{n-r}2^n\frac{(n-r)!\,r!}{(2n-2r)!}P_n^{n-2r}(\tanh\tfrac{1}{2}x).$$

This is easily verified by showing that the two sides satisfy the same differential equation, and are equal, with their first derivatives, at $x = 0$.

There are also poles of $m_2(\lambda)$ at the points

$$-i\surd\lambda = \tfrac{1}{2}n-\tfrac{1}{2}-r \quad (r = 0, 1,..., [\tfrac{1}{2}n-\tfrac{1}{2}]),$$

the residues being

$$B_r = \frac{(-1)^r}{r!}\frac{(2r-n+1)\Gamma(n+\frac{1}{2}-r)}{\Gamma(-\frac{1}{2}-r)\Gamma(n-r)}.$$

This is case (iii) of § 2.18, with $\mu_2 = B_r$, $\mu_1 = -B_r$. Hence $\Phi(x,\lambda)$ has the residue

$$\tfrac{1}{2}B_r\,\phi(x,\lambda_r')\int\limits_{-\infty}^{\infty}\phi(y,\lambda_r')f(y)\,dy,$$

where

$$\phi(x,\lambda_r') = -2\cosh^{n+1}\tfrac{1}{2}x\sinh\tfrac{1}{2}x\,F(r+\tfrac{3}{2},n-r+\tfrac{1}{2},\tfrac{3}{2},-\sinh^2\tfrac{1}{2}x).$$

It is easily verified as before that this is equal to

$$(-1)^{n-r}\frac{2^n(n-r)!\,r!}{(2n-2r)!}P_n^{n-2r-1}(\tanh\tfrac{1}{2}x).$$

Another interesting particular case occurs when the coefficient of $\cosh x$ in (4.17.4) vanishes. The solutions are then expressible in terms of associated Legendre functions with argument $\coth\frac{1}{2}x$.

4.19. Another formula involving hypergeometric functions arises as follows. Consider the equation

$$X(1+X)\frac{d^2Y}{dX^2}+\{c+(a+b+1)X\}\frac{dY}{dX}+abY = 0 \qquad (4.19.1)$$

satisfied by $Y = F(a,b,c,-X)$. Putting

$$Y_1 = X^{\frac{1}{2}c}(1+X)^{\frac{1}{2}(a+b+1-c)}Y$$

we obtain†

$$\frac{d^2Y_1}{dX^2}+ZY_1 = 0,$$

† See Forsyth, *A Treatise on Differential Equations*, (4th ed.), § 116.

where

$$Z = \frac{1-(1-c)^2}{4X^2}+\frac{1-(c-a-b)^2}{4(1+X)^2}+\frac{(1-c)^2-(a-b)^2+(c-a-b)^2-1}{4X(1+X)}.$$

Putting $y = X^{-\frac{1}{2}}Y_1$, $X = e^x$, this gives

$$\frac{d^2y}{dx^2}+(X^2Z+\tfrac{1}{4})y = 0. \tag{4.19.2}$$

Now $$X^2Z-\tfrac{1}{4} = \lambda-\frac{AX}{X+1}-\frac{BX}{(X+1)^2} = \lambda-q(x),$$

where $$A = \tfrac{1}{4}(a-b)^2-\tfrac{1}{4}(1-c)^2, \qquad B = \tfrac{1}{4}-\tfrac{1}{4}(c-a-b)^2,$$
$$\lambda = -\tfrac{1}{4}(1-c)^2.$$

Writing $1-4B = (1-2\alpha)^2$, the relations between the parameters are $a = \alpha+i\surd\lambda+i\surd(\lambda-A)$, $\quad b = \alpha+i\surd\lambda-i\surd(\lambda-A)$, $\quad c = 1+2i\surd\lambda$. Solutions of (4.19.2) which are $L^2(-\infty,0)$ and $L^2(0,\infty)$ respectively for $\mathbf{I}(\lambda) > 0$ are

$$\phi_1(x) = X^{\frac{1}{2}c-\frac{1}{2}}(1+X)^{\frac{1}{2}(a+b+1-c)}F(a,b,c,-X)$$

and

$$\phi_2(x) = X^{\frac{1}{2}c-\frac{1}{2}-b}(1+X)^{\frac{1}{2}(a+b+1-c)}F(b,1-c+b,1-a+b,-1/X).$$

The resulting expansion formula is of the type (3.1.8), but does not seem to reduce to a very simple form.

If $A = 0$, the $q(x)$ of this section is the same as that of § 4.18. The two sets of formulae are connected by the relation†

$$(1+z)^{2P}\, F(2P,2P+1-Q,Q,z) = F\{P,P+\tfrac{1}{2},Q,4z/(1+z)^2\}.$$

REFERENCES

§ 4.2. Titchmarsh (4).

§§ 4.3–4.7. Titchmarsh (3). Naturally these formulae are treated by Szegö in more detail.

§§ 4.8–4.9. See Watson, *Theory of Bessel Functions*, ch. xviii.

§ 4.10. Titchmarsh (1).

§ 4.11. Weyl (3), Watson, *Theory of Bessel Functions*, ch. xiv. So far as I know, the case $0 \leqslant \nu < 1$ has not previously been worked out in detail.

§ 4.12. Bell (1).

§ 4.13. Weyl (2), Titchmarsh (3).

§ 4.14. This example was pointed out to me by Mr. Crum.

§ 4.15. Sonine (1) 41–2, Burnett (1).

§ 4.16. Schrödinger (1), Titchmarsh (3).

§ 4.17–18. Weyl (1); information regarding P_n^m supplied by Miss K. Sarginson.

§ 4.19. Eckart (1).

† Forsyth, ibid. 241, 5 (i).

V

THE NATURE OF THE SPECTRUM

5.1. The main object of this chapter is to determine how the spectrum depends on the function $q(x)$.

Take the interval to be $(0,\infty)$, with no singularity except at infinity. Then there are, broadly speaking, four different cases. If $q(x) \to \infty$, there is purely a point-spectrum; the examples $q(x) = x^2$, $q(x) = x$ of § 4.2 and § 4.12 illustrate this. If $q(x) \to 0$, there is a continuous spectrum in $(0,\infty)$, with a point-spectrum (which may be null) in $(-\infty, 0)$; the formulae of § 4.10 and § 4.17 are examples. If $q(x) \to -\infty$, but so that $\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx$ is divergent, the spectrum extends continuously from $-\infty$ to ∞, as in the formula of § 4.13. Lastly, if $q(x) \to -\infty$, and $\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx$ is convergent, there is a continuous spectrum in $(-\infty, 0)$, and a point-spectrum (which may be null) in $(0,\infty)$. This is illustrated by § 4.14.

Actually in each case, except the first, we have to impose other conditions, so that classification is by no means complete; but all ordinary examples come under one or other of our theorems.

5.2. We begin with the second of the above cases, but actually assume, instead of $q(x) \to 0$, that $q(x)$ is $L(0,\infty)$.

We require the following

LEMMA 5.2. *Let $f(x) \geqslant 0$, $g(x) \geqslant 0$, and let $f(x)$ be continuous, $g(x)$ integrable, in $0 \leqslant x \leqslant X$. Let*

$$f(x) \leqslant C + \int_0^x f(t)g(t)\, dt \quad (0 \leqslant x \leqslant X). \tag{5.2.1}$$

Then
$$f(x) \leqslant Ce^{\int_0^x g(t)\, dt} \quad (0 \leqslant x \leqslant X). \tag{5.2.2}$$

Let

$$y = \int_0^x f(t)g(t)\, dt,$$

$$\frac{dy}{dx} = f(x)g(x).$$

Multiplying (5.2.1) by $g(x)$,

$$\frac{dy}{dx} \leqslant Cg(x)+yg(x),$$

$$\frac{d}{dx}\left\{ye^{-\int\limits_0^x g(t)\,dt}\right\} \leqslant Cg(x)e^{-\int\limits_0^x g(t)\,dt}.$$

Integrating over $(0,x)$,

$$ye^{-\int\limits_0^x g(t)\,dt} \leqslant C\left\{1-e^{-\int\limits_0^x g(t)\,dt}\right\}, \qquad y \leqslant C\left\{e^{\int\limits_0^x g(t)\,dt}-1\right\},$$

and (5.2.2) follows.

5.3. Let $\phi(x) = \phi(x,\lambda)$ be the solution of

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0$$

with $\phi(0) = \sin\alpha$, $\phi'(0) = -\cos\alpha$. By (1.7.1), with $a = 0$, $\lambda = s^2$,

$$\phi(x) = \cos sx\sin\alpha-\frac{\sin sx}{s}\cos\alpha+ \\ +\frac{1}{s}\int\limits_0^x \sin\{s(x-y)\}q(y)\phi(y)\,dy. \qquad (5.3.1)$$

Let $s = \sigma+it$, $t \geqslant 0$, and write temporarily $\phi_1(x) = \phi(x)e^{-tx}$. Then

$$\phi_1(x) = e^{-tx}\cos sx\sin\alpha-e^{-tx}\frac{\sin sx}{s}\cos\alpha+ \\ +\frac{1}{s}\int\limits_0^x e^{-t(x-y)}\sin\{s(x-y)\}q(y)\phi_1(y)\,dy.$$

Since $|\cos sx| \leqslant e^{tx}$, $|\sin sx| \leqslant e^{tx}$, it follows that

$$|\phi_1(x)| \leqslant 1+\frac{1}{|s|}+\frac{1}{|s|}\int\limits_0^x |q(y)\phi_1(y)|\,dy.$$

Hence, by the lemma,

$$|\phi_1(x)| \leqslant \left(1+\frac{1}{|s|}\right)\exp\left\{\frac{1}{|s|}\int\limits_0^x |q(y)|\,dy\right\}.$$

Since $q(y)$ is $L(0,\infty)$, it follows that $\phi_1(x)$ is bounded for all x, $|s| \geqslant \rho > 0$, $t \geqslant 0$.

Now consider real positive values of s. Then $\phi(x)$ is bounded for $s \geqslant \rho$. Hence (5.3.1) gives

$$\phi(x) = \cos sx \sin\alpha - \frac{\sin sx}{s}\cos\alpha + $$
$$+\frac{1}{s}\int_0^\infty \sin s(x-y)q(y)\phi(y)\,dy + O\left\{\int_x^\infty |q(y)|\,dy\right\}$$
$$= \mu(\lambda)\cos sx + \nu(\lambda)\sin sx + o(1) \tag{5.3.2}$$

(as $x \to \infty$), where

$$\mu(\lambda) = \mu(s^2) = \sin\alpha - \frac{1}{s}\int_0^\infty \sin sy\, q(y)\phi(y)\,dy,$$

$$\nu(\lambda) = -\frac{\cos\alpha}{s} + \frac{1}{s}\int_0^\infty \cos sy\, q(y)\phi(y)\,dy.$$

Since the integrals converge uniformly, μ and ν are continuous functions of s.

Similarly if $\theta(0) = \cos\alpha$, $\theta'(0) = \sin\alpha$,

$$\theta(x) = \mu_1(\lambda)\cos sx + \nu_1(\lambda)\sin sx + o(1), \tag{5.3.3}$$

where

$$\mu_1(\lambda) = \cos\alpha - \frac{1}{s}\int_0^\infty \sin sy\, q(y)\theta(y)\,dy,$$

$$\nu_1(\lambda) = \frac{\sin\alpha}{s} + \frac{1}{s}\int_0^\infty \cos sy\, q(y)\phi(y)\,dy.$$

Also, differentiating (5.3.1),

$$\phi'(x) = -s\sin sx \sin\alpha - \cos sx \cos\alpha + \int_0^\infty \cos s(x-y)q(y)\phi(y)\,dy.$$

Applying this in a similar way, we obtain

$$\phi'(x) = -s\mu(\lambda)\sin sx + s\nu(\lambda)\cos sx + o(1).$$

Similarly

$$\theta'(x) = -s\mu_1(\lambda)\sin sx + s\nu_1(\lambda)\cos sx + o(1).$$

Hence

$$W(\phi, \theta) = W\{\mu(\lambda)\cos sx + \nu(\lambda)\sin sx,\ \mu_1(\lambda)\cos sx + \nu_1(\lambda)\sin sx\} + o(1)$$
$$= s\{\mu(\lambda)\nu_1(\lambda) - \mu_1(\lambda)\nu(\lambda)\} + o(1).$$

Since $W(\phi,\theta) = 1$, it follows that

$$\mu(\lambda)\nu_1(\lambda)-\mu_1(\lambda)\nu(\lambda) = \frac{1}{\sqrt{\lambda}}. \tag{5.3.4}$$

Now consider complex values of s. For a fixed positive t, (5.3.1) gives

$$\phi(x) = \tfrac{1}{2}e^{-isx}\sin\alpha+\frac{e^{-isx}}{2is}\cos\alpha+O(e^{-tx})-$$
$$-\frac{1}{2is}\int_0^x e^{-is(x-y)}q(y)\phi(y)\,dy+O\left\{\int_0^x e^{-t(x-y)}|q(y)\phi(y)|\,dy\right\}$$

as $x\to\infty$. Since $\phi(y) = O(e^{ty})$, the last term is

$$O\left\{\int_0^x e^{t(2y-x)}|q(y)|\,dy\right\}$$
$$= O\left\{e^{t(x-2\delta)}\int_0^{x-\delta}|q(y)|\,dy\right\}+O\left\{e^{tx}\int_{x-\delta}^{x}|q(y)|\,dy\right\} = o(e^{tx}).$$

Also $$\int_x^\infty e^{-is(x-y)}q(y)\phi(y)\,dy = O\left\{e^{tx}\int_x^\infty |q(y)|\,dy\right\} = o(e^{tx}).$$

Hence $$\phi(x) = e^{-isx}\{M(\lambda)+o(1)\}, \tag{5.3.5}$$

where $$M(\lambda) = \tfrac{1}{2}\sin\alpha+\frac{\cos\alpha}{2is}-\frac{1}{2is}\int_0^\infty e^{isy}q(y)\phi(y)\,dy. \tag{5.3.6}$$

Similarly $$\theta(x) = e^{-isx}\{M_1(\lambda)+o(1)\}, \tag{5.3.7}$$

where $$M_1(\lambda) = \tfrac{1}{2}\cos\alpha-\frac{\sin\alpha}{2is}-\frac{1}{2is}\int_0^\infty e^{isy}q(y)\theta(y)\,dy. \tag{5.3.8}$$

Since $\theta(x)+m(\lambda)\phi(x)$ is $L^2(0,\infty)$, it follows that

$$m(\lambda) = -M_1(\lambda)/M(\lambda). \tag{5.3.9}$$

As s tends to a real limit, the numerator and denominator in (5.3.9) tend to $\frac{1}{2}\mu_1(\lambda)+\frac{1}{2}i\nu_1(\lambda)$ and $\frac{1}{2}\mu(\lambda)+\frac{1}{2}i\nu(\lambda)$ respectively. By (5.3.4), $\mu(\lambda)$ and $\nu(\lambda)$ cannot both vanish for any positive λ. Hence

$$\lim m(\lambda) = -\frac{\mu_1(\lambda)+i\nu_1(\lambda)}{\mu(\lambda)+i\nu(\lambda)},$$

and the imaginary part of this is

$$-\frac{\mu(\lambda)\nu_1(\lambda)-\mu_1(\lambda)\nu(\lambda)}{\mu^2(\lambda)+\nu^2(\lambda)} = -\frac{1}{\lambda^{\frac{1}{2}}\{\mu^2(\lambda)+\nu^2(\lambda)\}}.$$

Hence in the notation of § 3.3

$$k(\lambda) = \int_0^\lambda \frac{du}{u^{\frac{1}{2}}\{\mu^2(u)+\nu^2(u)\}}. \tag{5.3.10}$$

Hence the spectrum is continuous in $(0,\infty)$. The contribution of this part to (3.1.1) is

$$\frac{1}{\pi}\int_0^\infty \frac{\phi(x,\lambda)}{\lambda^{\frac{1}{2}}\{\mu^2(\lambda)+\nu^2(\lambda)\}}\,d\lambda \int_0^\infty \phi(y,\lambda)f(y)\,dy$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\phi(x,s^2)}{\mu^2(s^2)+\nu^2(s^2)}\,ds \int_0^\infty \phi(y,s^2)f(y)\,dy.$$

Next let $s \to it$, where t is real and positive. The integrals in (5.3.6) and (5.3.8) converge uniformly, and hence $M(\lambda)$ and $M_1(\lambda)$ tend to the real limits

$$\tfrac{1}{2}\sin\alpha - \frac{\cos\alpha}{2t} + \frac{1}{2t}\int_0^\infty e^{-ty}q(y)\phi(y)\,dy,$$

$$\tfrac{1}{2}\cos\alpha + \frac{\sin\alpha}{2t} + \frac{1}{2t}\int_0^\infty e^{-ty}q(y)\theta(y)\,dy.$$

Hence $\mathbf{I}(m) \to 0$ except possibly at the zeros of $M(\lambda)$. Since $M(\lambda)$ is an analytic function of s, regular for $\mathbf{I}(s) > 0$, the zeros are isolated points. Hence there is a point-spectrum in $-\infty < \lambda < 0$, which is bounded below, since $M(\lambda) \sim \frac{1}{2}\sin\alpha$ or $\cos\alpha/2is$ as $s \to \infty$.

5.4. A transformation of the basic equation. In order to deal with cases in which $q(x)$ is large at infinity, we make a transformation of the equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0 \tag{5.4.1}$$

which will be used frequently in later sections. Suppose first that λ is real, $q(x) < \lambda$, and let $q'(x)$ and $q''(x)$ be continuous. Let

$$\xi(x) = \int_0^x \{\lambda - q(t)\}^{\frac{1}{2}}\,dt, \qquad \eta(x) = \{\lambda - q(x)\}^{\frac{1}{4}}y.$$

Then

$$\frac{d\eta}{d\xi} = \frac{d\eta}{dx}\frac{dx}{d\xi} = \left[\{\lambda-q(x)\}^{\frac{1}{4}}\frac{dy}{dx}-\frac{1}{4}\frac{q'(x)}{\{\lambda-q(x)\}^{\frac{3}{4}}}y\right]\frac{1}{\{\lambda-q(x)\}^{\frac{1}{2}}}$$

$$= \{\lambda-q(x)\}^{-\frac{1}{4}}\frac{dy}{dx}-\frac{1}{4}\frac{q'(x)}{\{\lambda-q(x)\}^{\frac{5}{4}}}y,$$

$$\frac{d^2\eta}{d\xi^2} = \left[\{\lambda-q(x)\}^{-\frac{1}{4}}\frac{d^2y}{dx^2}-\frac{1}{4}\left\{\frac{q''(x)}{\{\lambda-q(x)\}^{\frac{5}{4}}}+\frac{5}{4}\frac{q'^2(x)}{\{\lambda-q(x)\}^{\frac{9}{4}}}\right\}y\right]\frac{1}{\{\lambda-q(x)\}^{\frac{1}{2}}}.$$

Hence (5.4.1) transforms into

$$\frac{d^2\eta}{d\xi^2}+\eta+\left[\frac{1}{4}\frac{q''(x)}{\{\lambda-q(x)\}^2}+\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^3}\right]\eta = 0. \tag{5.4.2}$$

This is an equation of the same form as (5.4.1); but in (5.4.2) the coefficient of η in the last term is in general small when λ is large, or when $q(x)$ is large and negative.

It follows that, if $\eta = \phi(\xi)$, ϕ satisfies the integral equation

$$\phi(\xi) = \phi(0)\cos\xi+\phi'(0)\sin\xi-$$

$$-\int\limits_0^{\xi}\sin(\xi-\tau)\left[\frac{1}{4}\frac{q''(t)}{\{\lambda-q(t)\}^2}+\frac{5}{16}\frac{q'^2(t)}{\{\lambda-q(t)\}^3}\right]\phi(\tau)\,d\tau, \tag{5.4.3}$$

where $\tau = \xi(t)$. This can be used as in previous sections to obtain asymptotic formulae for $\phi(\xi)$.

If λ is not real, or $q(x) > \lambda$, ξ is not real, and the above formulae would involve integrals along complex paths. It is not necessary to introduce such integrals, since we can obtain the corresponding integral equation in terms of the real variable x directly, as follows.

Let $q(0) = 0$ (this involves at most a change in the λ-origin), and write

$$P(x) = \{\lambda-q(x)\}^{\frac{1}{4}}\frac{d}{dx}\left[\{\lambda-q(x)\}^{-\frac{1}{2}}\frac{d\eta}{dx}\right]-\frac{d^2y}{dx^2}. \tag{5.4.4}$$

Then

$$P(x) = \{\lambda-q(x)\}^{\frac{1}{4}}\frac{d}{dx}\left[\{\lambda-q(x)\}^{-\frac{1}{4}}\frac{dy}{dx}-\frac{1}{4}\frac{q'(x)}{\{\lambda-q(x)\}^{\frac{5}{4}}}y\right]-\frac{d^2y}{dx^2}$$

$$= -\left[\frac{1}{4}\frac{q''(x)}{\{\lambda-q(x)\}}+\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}\right]y.$$

Let

$$I = \int_0^x \sin\{\xi(x)-\xi(t)\}\frac{P(t)}{\{\lambda-q(t)\}^{\frac{1}{4}}}\,dt$$

$$= \int_0^x \sin\{\xi(x)-\xi(t)\}\frac{d}{dt}\left[\{\lambda-q(t)\}^{-\frac{1}{4}}\frac{d\eta}{dt}\right]dt + $$

$$+\int_0^x \sin\{\xi(x)-\xi(t)\}\{\lambda-q(t)\}^{\frac{1}{4}}y\,dt$$

$$= I_1+I_2$$

by (5.4.4) and (5.4.1). Now

$$I_1 = \left[\sin\{\xi(x)-\xi(t)\}\{\lambda-q(t)\}^{-\frac{1}{4}}\frac{d\eta}{dt}\right]_0^x + \int_0^x \cos\{\xi(x)-\xi(t)\}\frac{d\eta}{dt}\,dt$$

$$= -\eta'(0)\lambda^{-\frac{1}{4}}\sin\xi(x)+[\cos\{\xi(x)-\xi(t)\}\eta(t)]_0^x-$$

$$-\int_0^x \sin\{\xi(x)-\xi(t)\}\{\lambda-q(t)\}^{\frac{1}{4}}\eta(t)\,dt$$

$$= -\eta'(0)\lambda^{-\frac{1}{4}}\sin\xi(x)+\eta(x)-\eta(0)\cos\xi(x)-I_2.$$

Hence $$\eta(x) = \eta(0)\cos\xi(x)+\eta'(0)\lambda^{-\frac{1}{4}}\sin\xi(x)+I,$$

i.e. $\eta(x)$ satisfies the integral equation

$$\eta(x) = \eta(0)\cos\xi(x)+\eta'(0)\lambda^{-\frac{1}{4}}\sin\xi(x)+$$

$$+\int_0^x \sin\{\xi(x)-\xi(t)\}R(t)\eta(t)\,dt, \quad (5.4.5)$$

where $$R(t) = -\frac{1}{4}\frac{q''(t)}{\{\lambda-q(t)\}^{\frac{3}{4}}}-\frac{5}{16}\frac{q'^2(t)}{\{\lambda-q(t)\}^{\frac{7}{4}}}. \quad (5.4.6)$$

In these formulae, if $\mathbf{I}(\lambda) > 0$, we take $0 < \arg\lambda < \pi$, and, if e.g. $q(t)$ varies from 0 to ∞, $\arg\{\lambda-q(t)\}$ varies from $\arg\lambda$ to π.

5.5. The case $q(x) \to \infty$. We impose the following additional conditions. *Let $q(x)$ tend steadily to infinity, so that $q'(x) \geqslant 0$; let*

$$q'(x) = O[\{q(x)\}^c] \quad (0 < c < \tfrac{3}{2}), \quad (5.5.1)$$

and let $q''(x)$ be ultimately of one sign. Then

$$\int_{X_0}^{X} \frac{q'^2(x)}{\{q(x)\}^{\frac{5}{2}}}\,dx = O\left[\int_{X_0}^{X} \frac{q'(x)}{\{q(x)\}^{\frac{5}{2}-c}}\,dx\right]$$

$$= O[\{q(x)\}^{c-\frac{3}{2}}]_{X_0}^{X} = O(1),$$

and hence also

$$\int\limits_{X_0}^{X} \frac{q''(x)}{\{q(x)\}^{\frac{3}{2}}}\,dx = \left[\frac{q'(x)}{\{q(x)\}^{\frac{3}{2}}}\right]_{X_0}^{X} + \frac{3}{2}\int\limits_{X_0}^{X} \frac{q'^2(x)}{\{q(x)\}^{\frac{5}{2}}}\,dx = O(1).$$

Hence
$$\int\limits_{0}^{\infty} |R(x)|\,dx \tag{5.5.2}$$

is convergent, uniformly with respect to λ over any region for which $|\lambda - q(x)| \geqslant \delta > 0$ for $0 \leqslant x < \infty$.

5.6. THEOREM 5.6. *If $q(x)$ satisfies the conditions of § 5.5, the spectrum is discrete.*

Since
$$\tfrac{1}{2}\arg\lambda \leqslant \arg\{\lambda - q(t)\}^{\frac{1}{2}} \leqslant \tfrac{1}{2}\pi,$$
the imaginary part of $\xi(x)$ is positive if $\mathbf{I}(\lambda) > 0$; as $x \to \infty$, while λ remains bounded,

$$\xi(x) \sim i\int\limits_{0}^{x} \{q(t)\}^{\frac{1}{2}}\,dt,$$

and so $e^{-i\xi(x)} \to \infty$.

We can now argue with (5.4.5), as we did previously with (5.3.1) to obtain (5.3.5). Let $\eta_1(x) = e^{i\xi(x)}\eta(x)$. Then (5.4.5) gives

$$\eta_1(x) = \eta(0)e^{i\xi(x)}\cos\xi(x) + \eta'(0)\lambda^{-\frac{1}{2}}e^{i\xi(x)}\sin\xi(x) +$$
$$+ \int\limits_{0}^{x} e^{i\{\xi(x)-\xi(t)\}}\sin\{\xi(x)-\xi(t)\}R(t)\eta_1(t)\,dt.$$

Hence
$$|\eta_1(x)| \leqslant |\eta(0)| + |\eta'(0)\lambda^{-\frac{1}{2}}| + \int\limits_{0}^{x} |R(t)\eta_1(t)|\,dt,$$

so that by Lemma 5.2

$$|\eta_1(x)| \leqslant \{|\eta(0)| + |\eta'(0)\lambda^{-\frac{1}{2}}|\}\exp\left\{\int\limits_{0}^{x} |R(t)|\,dt\right\}$$
$$\leqslant \{|\eta(0)| + |\eta'(0)\lambda^{-\frac{1}{2}}|\}\exp\left\{\int\limits_{0}^{\infty} |R(t)|\,dt\right\}.$$

Hence

$$|\eta(x)| \leqslant \{|\eta(0)| + |\eta'(0)\lambda^{-\frac{1}{2}}|\}\exp\left\{\int\limits_{0}^{\infty} |R(t)|\,dt\right\}|e^{-i\xi(x)}|.$$

Applying this to (5.4.5), we conclude as before that, for any fixed λ in the upper half-plane, as $x \to \infty$

$$\eta(x) \sim \left\{\tfrac{1}{2}\eta(0) + \tfrac{1}{2}i\eta'(0)\lambda^{-\frac{1}{2}} + \tfrac{1}{2}i\int\limits_{0}^{\infty} e^{i\xi(t)}R(t)\eta(t)\,dt\right\}e^{-i\xi(x)}.$$

Now let $\eta(x) = \{\lambda - q(x)\}^{\frac{1}{4}}\phi(x,\lambda)$. Then

$$\eta(0) = \lambda^{\frac{1}{4}}\sin\alpha, \qquad \eta'(0) = -\lambda^{\frac{1}{4}}\cos\alpha - \frac{1}{4}\frac{q'(0)}{\lambda^{\frac{3}{4}}}\sin\alpha.$$

Hence
$$|\phi(x,\lambda)| \leqslant O(\lambda^{\frac{1}{4}})\exp\left\{\int\limits_0^\infty |R(t)|\,dt\right\}\frac{|e^{-i\xi(x)}|}{|\lambda - q(x)|^{\frac{1}{4}}} \tag{5.6.1}$$

for all x and $\mathbf{I}(\lambda) > 0$; and for a fixed λ, as $x \to \infty$

$$\phi(x,\lambda) \sim \frac{M(\lambda)e^{-i\xi(x)}}{\{\lambda - q(x)\}^{\frac{1}{4}}}, \tag{5.6.2}$$

where

$$M(\lambda) = \tfrac{1}{2}\lambda^{\frac{1}{4}}\sin\alpha - \tfrac{1}{2}i\left\{\frac{\cos\alpha}{\lambda^{\frac{1}{4}}} + \frac{q'(0)\sin\alpha}{4\lambda^{\frac{5}{4}}}\right\} +$$
$$+\tfrac{1}{2}i\int\limits_0^\infty e^{i\xi(t)}R(t)\{\lambda - q(t)\}^{\frac{1}{4}}\phi(t)\,dt.$$

Similarly
$$\theta(x,\lambda) \sim \frac{M_1(\lambda)e^{-i\xi(x)}}{\{\lambda - q(x)\}^{\frac{1}{4}}},$$

where $M_1(\lambda)$ is obtained from $M(\lambda)$ by replacing $\sin\alpha$, $-\cos\alpha$, $\phi(t)$ by $\cos\alpha$, $\sin\alpha$, $\theta(t)$. Hence

$$m(\lambda) = -M_1(\lambda)/M(\lambda).$$

Now it clearly follows from (5.4.5) and Lemma 5.2 (as in § 5.3) that $\eta(x) = O\{|e^{-i\xi(x)}|\}$ uniformly with respect to λ as λ approaches any interval of the negative real axis. Hence $\lambda^{-\frac{1}{4}}M(\lambda)$ is continuous up to the negative real axis, and it is also obviously real there, $\xi(t)$ and $R(t)$ being purely imaginary.

If we do not choose $q(0) = 0$, the argument shows that

$$\{\lambda - q(0)\}^{-\frac{1}{4}}M(\lambda)$$

is real and continuous for $\lambda < q(0)$. But the whole argument could be constructed equally well with an interval (X,∞) instead of $(0,\infty)$. We should then obtain an alternative expression for $M(\lambda)$ involving an integral over (X,∞). Hence $\{\lambda - q(X)\}^{-\frac{1}{4}}M(\lambda)$ is real and continuous for $\lambda < q(X)$; hence in fact $e^{-\frac{1}{4}i\pi}M(\lambda)$ is real and continuous along the whole real axis. Since $M(\lambda)$ is regular in the upper half-plane, it follows from the principle of reflection that $M(\lambda)$ is an integral function. Similarly $M_1(\lambda)$ is an integral function. Hence $m(\lambda)$ is meromorphic, and the result follows.

It will be shown later that the result holds if $q(x) \to \infty$, without

any other restrictions. This depends on the theory of the zeros of eigenfunctions given at the end of this chapter.

5.7. THEOREM 5.7. *Let $q(x) \leqslant 0$, $q'(x) < 0$, $q(x) \to -\infty$,*

$$q'(x) = O\{|q(x)|^c\} \quad (0 < c < \tfrac{3}{2}) \tag{5.7.1}$$

and let $q''(x)$ be ultimately of one sign.

Then if

$$\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx \tag{5.7.2}$$

is divergent, there is a continuous spectrum over $(-\infty, \infty)$.

The conditions imply the convergence of (5.5.2).

For λ real and positive, (5.4.5) now gives

$$\eta(x) = \eta(0)\cos\xi(x) + \eta'(0)\lambda^{-\frac{1}{2}}\sin\xi(x) + {}$$
$$+ \int_0^{\infty} \sin\{\xi(x) - \xi(t)\}R(t)\eta(t)\, dt + o(1).$$

Hence

$$\phi(x,\lambda) = \{\lambda - q(x)\}^{-\frac{1}{4}}\{\mu(\lambda)\cos\xi(x) + \nu(\lambda)\sin\xi(x) + o(1)\}, \tag{5.7.3}$$

where

$$\mu(\lambda) = \lambda^{\frac{1}{2}}\sin\alpha - \int_0^{\infty} \sin\xi(t)R(t)\{\lambda - q(t)\}^{\frac{1}{4}}\phi(t,\lambda)\, dt, \tag{5.7.4}$$

$$\nu(\lambda) = -\frac{q'(0)\sin\alpha}{4\lambda^{\frac{3}{4}}} - \frac{\cos\alpha}{\lambda^{\frac{1}{4}}} + \int_0^{\infty} \cos\xi(t)R(t)\{\lambda - q(t)\}^{\frac{1}{4}}\phi(t,\lambda)\, dt. \tag{5.7.5}$$

Similarly, if $\theta(x,\lambda)$ is the solution of (5.4.1) such that $\theta(0,\lambda) = \cos\alpha$, $\theta'(0,\lambda) = \sin\alpha$, then

$$\theta(x,\lambda) = \{\lambda - q(x)\}^{-\frac{1}{4}}\{\mu_1(\lambda)\cos\xi(x) + \nu_1(\lambda)\sin\xi(x) + o(1)\}, \tag{5.7.6}$$

where μ_1 and ν_1 are obtained from μ and ν by replacing $\sin\alpha$, $-\cos\alpha$, and ϕ by $\cos\alpha$, $\sin\alpha$, and θ.

The argument also shows that the integrals in (5.7.4), etc., converge uniformly with respect to λ, and hence that $\mu(\lambda)$, etc., are continuous functions of λ.

Again, differentiating (5.4.5),

$$\eta'(x) = \{\lambda - q(x)\}^{\frac{1}{2}}\Big[\lambda^{-\frac{1}{2}}\eta'(0)\cos\xi(x) - \eta(0)\sin\xi(x) + {}$$
$$+ \int_0^{x} \cos\{\xi(x) - \xi(t)\}R(t)\eta(t)\, dt\Big].$$

Applying this in a similar way, we obtain

$$\frac{d}{dx}[\{\lambda-q(x)\}^{\frac{1}{4}}\phi(x,\lambda)] \sim \{\lambda-q(x)\}^{\frac{1}{4}}\{\nu(\lambda)\cos\xi(x)-\mu(\lambda)\sin\xi(x)\},$$

whence by (5.7.3) and (5.7.1)

$$\phi'(x,\lambda) \sim \{\lambda-q(x)\}^{\frac{1}{4}}\{\nu(\lambda)\cos\xi(x)-\mu(\lambda)\sin\xi(x)\}. \tag{5.7.7}$$

Similarly

$$\theta'(x,\lambda) \sim \{\lambda-q(x)\}^{\frac{1}{4}}\{\nu_1(\lambda)\cos\xi(x)-\mu_1(\lambda)\sin\xi(x)\}. \tag{5.7.8}$$

Hence $$\lim_{x\to\infty} W(\phi,\theta) = \mu(\lambda)\nu_1(\lambda)-\mu_1(\lambda)\nu(\lambda).$$

Since $W(\phi,\theta) = 1$, it follows that

$$\mu(\lambda)\nu_1(\lambda)-\mu_1(\lambda)\nu(\lambda) = 1. \tag{5.7.9}$$

Hence $\mu(\lambda)$ and $\nu(\lambda)$ do not both vanish for any value of λ.

The argument requires modification if $\lambda < 0$; but then we can choose X so that $\lambda-q(x) > 0$ for $x \geqslant X$, and we can apply the whole argument to the interval (X,∞) instead of to $(0,\infty)$. The same conclusion then follows.

Now let $\lambda = u+iv$, where $v > 0$. Taking x_0 so that $u-q(t) > v$ in $x > x_0$, we have

$$\xi(x) = \int_0^x \{u+iv-q(t)\}^{\frac{1}{2}}\,dt = \xi_0+\int_{x_0}^x \{u+iv-q(t)\}^{\frac{1}{2}}\,dt$$

$$= \xi_0+\int_{x_0}^x \{u-q(t)\}^{\frac{1}{2}}\,dt + \tfrac{1}{2}iv\int_{x_0}^x \{u-q(t)\}^{-\frac{1}{2}}\,dt + O\left\{v^2\int_{x_0}^x |q(t)|^{-\frac{3}{2}}\,dt\right\}.$$

Hence as $x \to \infty$

$$\mathbf{I}(\xi) \sim \tfrac{1}{2}v\int_{x_0}^x \{u-q(t)\}^{-\frac{1}{2}}\,dt,$$

and so $e^{-i\xi(x)}$ is large for large x. Hence (5.4.5) gives

$$\eta(x) = \tfrac{1}{2}e^{-i\xi(x)}\left\{\eta(0)+i\lambda^{-\frac{1}{2}}\eta'(0)+i\int_0^x e^{i\xi(t)}R(t)\eta(t)\,dt + o(1)\right\}.$$

Proceeding as before, we obtain

$$\eta(x) \sim \tfrac{1}{2}e^{-i\xi(x)}\left\{\eta(0)+i\lambda^{-\frac{1}{2}}\eta'(0)+i\int_0^\infty e^{i\xi(t)}R(t)\eta(t)\,dt\right\}.$$

Hence $$\phi(x,\lambda) \sim M(\lambda)\{\lambda-q(x)\}^{-\frac{1}{4}}e^{-i\xi(x)},$$

where

$$M(\lambda) = \tfrac{1}{2}\lambda^{\frac{1}{2}}\sin\alpha - \tfrac{1}{2}i\left\{\frac{q'(0)\sin\alpha}{4\lambda^{\frac{1}{2}}} + \frac{\cos\alpha}{\lambda^{\frac{1}{2}}}\right\} +$$

$$+\tfrac{1}{2}i\int_0^\infty e^{i\xi(t)}R(t)\{\lambda-q(t)\}^{\frac{1}{2}}\phi(t,\lambda)\,dt,$$

and $$\theta(x,\lambda) \sim M_1(\lambda)\{\lambda-q(x)\}^{-\frac{1}{4}}e^{-i\xi(x)},$$

where $M_1(\lambda)$ is formed from $M(\lambda)$ by the same interchange as before.

According to the general theory of the solutions of (5.4.1), there is, for $v > 0$, a solution

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

which is $L^2(0,\infty)$. It follows from the above formulae that

$$m(\lambda) = -\frac{M_1(\lambda)}{M(\lambda)}.$$

Also, as $v \to 0$, $M(\lambda) \to \frac{1}{2}\mu(u)+\frac{1}{2}i\nu(u)$ and $M_1(\lambda) \to \frac{1}{2}\mu_1(u)+\frac{1}{2}i\nu_1(u)$, since it is easily seen that the integrals involved converge uniformly with respect to λ. Hence

$$\lim_{\nu\to 0} m(\lambda) = -\frac{\mu_1(u)+i\nu_1(u)}{\mu(u)+i\nu(u)} = -\frac{\{\mu_1(u)+i\nu_1(u)\}\{\mu(u)-i\nu(u)\}}{\mu^2(u)+\nu^2(u)},$$

$$\lim_{\nu\to 0}\{-\mathbf{I}m(\lambda)\} = \frac{\mu(u)\nu_1(u)-\mu_1(u)\nu(u)}{\mu^2(u)+\nu^2(u)} = \frac{1}{\mu^2(u)+\nu^2(u)}.$$

Since $k(\lambda)$ is the integral of this over $(0,\lambda)$, the result stated follows.

5.8. Theorem 5.8. *If the conditions of Theorem 5.7 are satisfied except that (5.7.2) is convergent, there is a continuous spectrum in $(-\infty,0)$ and a point-spectrum in $(0,\infty)$.*

As $x \to \infty$

$$\xi(x,\lambda)-\xi(x,0) = \int_0^x [\{\lambda-q(t)\}^{\frac{1}{2}}-\{-q(t)\}^{\frac{1}{2}}]\,dt$$

$$= \int_0^x \frac{\lambda\,dt}{\{\lambda-q(t)\}^{\frac{1}{2}}+\{-q(t)\}^{\frac{1}{2}}}$$

$$\to \int_0^\infty \frac{\lambda\,dt}{\{\lambda-q(t)\}^{\frac{1}{2}}+\{-q(t)\}^{\frac{1}{2}}} = \chi(\lambda) \qquad (5.8.1)$$

say. Hence $\mathbf{I}\xi(x,\lambda)$ is bounded, and so $\cos\xi(x)$ and $\sin\xi(x)$ are

bounded, whether λ is real or complex. It is then clear that the argument of § 4 holds, in this case, for all values of λ, real or complex. Thus (5.7.3), (5.7.6), (5.7.7), and (5.7.8) hold for all values of λ. In particular, all solutions of (5.4.1) are $L^2(0,\infty)$, and we are in Weyl's 'limit-circle' case, in which the expansion formula involves an arbitrary parameter.

Further $\mu(\lambda)$, $\nu(\lambda)$, $\mu_1(\lambda)$, and $\nu_1(\lambda)$ are integral functions of λ. For consider e.g. (5.7.4). Here $\{\lambda-q(t)\}^{\frac{1}{4}}$ is an analytic function of λ, regular except on the negative real axis; similarly for $\xi(t,\lambda)$ and $R(t)$, and the integral in (5.7.4) converges uniformly with respect to λ in any finite region, as the argument clearly shows. Hence $\mu(\lambda)$ is an analytic function, regular except possibly on the negative real axis. However, we could apply the argument leading to (5.7.3) equally well with any interval (X,∞), where $X > 0$, instead of the interval $(0,\infty)$. Hence we must also have

$$\mu(\lambda) = \{\lambda-q(X)\}^{\frac{1}{4}}\phi(X,\lambda)-\int\limits_X^\infty \sin\xi(t)R(t)\{\lambda-q(t)\}^{\frac{1}{4}}\phi(t,\lambda)\,dt.$$

Hence $\mu(\lambda)$ is regular except possibly on the real axis between $-\infty$ and $q(X)$; and so in fact it is an integral function. Similarly so are $\nu(\lambda)$, $\mu_1(\lambda)$, and $\nu_1(\lambda)$.

Now consider the solution $\theta(x,\lambda)+l(\lambda)\phi(x,\lambda)$ of (5.4.1) which satisfies the boundary condition

$$\{\theta(b)+l\phi(b)\}\cos\beta+\{\theta'(b)+l\phi'(b)\}\sin\beta = 0$$

at $x = b$. This gives

$$l(\lambda) = -\frac{\theta(b,\lambda)\cot\beta+\theta'(b,\lambda)}{\phi(b,\lambda)\cot\beta+\phi'(b,\lambda)}. \tag{5.8.2}$$

As $\cot\beta$ varies, $l(\lambda)$ describes a circle. According to Weyl's theory, this circle tends to a limit-circle as $b\to\infty$. Now by (5.7.3) and (5.7.7) the denominator in (5.8.2) is of the form

$$\{\lambda-q(b)\}^{-\frac{1}{4}}\{\mu(\lambda)\cos\xi(b)+\nu(\lambda)\sin\xi(b)+o(1)\}\cot\beta+$$
$$+\{\lambda-q(b)\}^{\frac{1}{4}}\{\nu(\lambda)\cos\xi(b)-\mu(\lambda)\sin\xi(b)+o(1)\},$$

or, since $$\{\lambda-q(b)\}^{-\frac{1}{4}} = \{-q(b)\}^{-\frac{1}{4}}\{1+o(1)\},$$

of the form

$$\{\lambda-q(b)\}^{\frac{1}{4}}[\{-q(b)\}^{-\frac{1}{2}}\{\mu(\lambda)\cos\xi(b)+\nu(\lambda)\sin\xi(b)+o(1)\}\cot\beta+$$
$$+\nu(\lambda)\cos\xi(b)-\mu(\lambda)\sin\xi(b)+o(1)].$$

Putting $\{-q(b)\}^{-\frac{1}{4}}\cot\beta = \cot\gamma$, (5.8.2) takes the form

$$l(\lambda) = -\frac{\mu_1(\lambda)\cos\{\xi(b)+\gamma\}+\nu_1(\lambda)\sin\{\xi(b)+\gamma\}+o\,(1)}{\mu(\lambda)\cos\{\xi(b)+\gamma\}+\nu_1(\lambda)\sin\{\xi(b)+\gamma\}+o\,(1)}.$$

Putting $\gamma = c-\xi(b,0)$, keeping c fixed and making $b \to \infty$, this tends to

$$m(\lambda) = -\frac{\mu_1(\lambda)\cos\{\chi(\lambda)+c\}+\nu_1(\lambda)\sin\{\chi(\lambda)+c\}}{\mu(\lambda)\cos\{\chi(\lambda)+c\}+\nu(\lambda)\sin\{\chi(\lambda)+c\}}.$$

As c varies this describes a circle, which is therefore Weyl's limit-circle.

For $\lambda > 0$, $\chi(\lambda)$ is real, and $\mathbf{I}m(\lambda) \to 0$ as λ tends to a real value, unless the above denominator vanishes. Hence the part of the spectrum on the positive real axis consists of the zeros of the denominator, which are isolated points since it is an analytic function.

On the other hand, if $\lambda < 0$, $\chi = \chi_1+i\chi_2$, where $\chi_2 > 0$; for if $q(h) = \lambda$, (5.8.1) gives

$$\chi_2 = \int_0^h \{q(x)-\lambda\}^{\frac{1}{2}}\,dx.$$

Hence, using (5.7.9), as λ tends to a real negative value, $-\mathbf{I}\{m(\lambda)\}$ tends to

$$\frac{\frac{1}{2}\sinh\{2\chi_2(\lambda)\}}{|\mu(\lambda)\cos\{\chi(\lambda)+c\}+\nu(\lambda)\sin\{\chi(\lambda)+c\}|^2},$$

which is finite and positive. Hence the spectrum is continuous from $-\infty$ to 0.

5.9. The zeros of eigenfunctions. This theory depends on the following fundamental theorem, due to Sturm.

Let u be a solution of

$$\frac{d^2u}{dx^2}+g(x)u = 0, \tag{5.9.1}$$

and v a solution of

$$\frac{d^2v}{dx^2}+h(x)v = 0, \tag{5.9.2}$$

where $g(x) < h(x)$ throughout the interval (a,b). Then between any two consecutive zeros of u there is at least one zero of v.

Multiplying by v, u respectively, and subtracting,

$$u''v-uv'' = \{h(x)-g(x)\}uv. \tag{5.9.3}$$

Let x_1, x_2 be consecutive zeros of u. Integrating from x_1 to x_2,

$$[u'v-uv']_{x_1}^{x_2} = \int_{x_1}^{x_2} \{h(x)-g(x)\}uv\, dx. \tag{5.9.4}$$

Suppose that v has no zero in (x_1, x_2). We may suppose without loss of generality that $u \geqslant 0$, $v \geqslant 0$ in (x_1, x_2). Then the right-hand side is positive (assuming that u and v do not vanish identically). Now the left-hand side is

$$u'(x_2)v(x_2)-u'(x_1)v(x_1)$$

and

$$u'(x_1) > 0, \qquad u'(x_2) < 0, \qquad v(x_1) \geqslant 0, \qquad v(x_2) \geqslant 0.$$

Hence the left-hand side is $\leqslant 0$, giving a contradiction.

5.10. We deduce the following theorem.

Let u be the solution of (5.9.1) *such that*

$$u(a) = \sin\alpha, \qquad u'(a) = -\cos\alpha,$$

and v the solution of (5.9.2) *such that*

$$v(a) = \sin\alpha, \qquad v'(a) = -\cos\alpha.$$

Then if $u(x)$ has m zeros in the interval $a < x \leqslant b$, $v(x)$ has at least m zeros in the same interval, and the νth zero of $v(x)$ is less than the νth zero of $u(x)$.

Suppose first that $\sin\alpha \neq 0$, say $\sin\alpha > 0$, so that the left-hand end-point is not a zero of either function.

In view of the previous theorem, it is only necessary to prove that $v(x)$ has at least one zero in the interval (a, x_1), where x_1 is the smallest zero of u. Now on integrating (5.9.3) from a to x_1, we obtain

$$u'(x_1)v(x_1) = \int_a^{x_1} \{h(x)-g(x)\}uv\, dx.$$

If $v(x)$ has no zero in (a, x_1), the right-hand side is positive; but

$$u'(x_1) < 0, \qquad v(x_1) > 0.$$

Hence we obtain a contradiction.

The result follows similarly if $\sin\alpha = 0$.

5.11. Now consider the eigenfunctions of Chapter I, defined as the solutions of

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0, \tag{5.11.1}$$

$$y(a)\cos\alpha+y'(a)\sin\alpha = 0, \tag{5.11.2}$$

$$y(b)\cos\beta+y'(b)\sin\beta = 0. \tag{5.11.3}$$

Consider first the solutions of (5.11.1) and (5.11.2), and suppose that $\sin\alpha \neq 0$. Then $y(a) \neq 0$. The number of zeros of $y(x)$ in (a,b) is a non-decreasing function of λ, by the above theorem. Let $|q(x)| < c$ in (a,b). First compare (5.11.1) with

$$\frac{d^2y_1}{dx^2}+(\lambda+c)y_1 = 0. \tag{5.11.4}$$

The solution of this satisfying (5.11.2) is

$$y_1 = \cosh\{(-\lambda-c)^{\frac{1}{2}}(x-a)\}-\frac{\cot\alpha}{(-\lambda-c)^{\frac{1}{2}}}\sinh\{(-\lambda-c)^{\frac{1}{2}}(x-a)\}.$$

This has no zeros if λ is negative and large enough. Hence the solution of (5.11.1) has no zeros in (a,b) if λ is negative and large enough.

Similarly, by considering the solution of

$$\frac{d^2y_2}{dx^2}+(\lambda-c)y_2 = 0$$

with λ large and positive, it follows that the number of zeros of (5.11.1) in (a,b) tends to infinity with λ.

Also, by the above theorems, a zero of y travels steadily to the left as λ increases. Hence there is an increasing sequence of numbers $\mu_0, \mu_1,\dots$ such that $y(b) = 0$ for $\lambda = \mu_m$, and $y(x)$ has just m zeros in $a < x < b$.

If $\sin\beta = 0$, the μ_m are the eigenvalues.

Otherwise, we have

$$\begin{aligned}\frac{d}{dx}\left\{u^2\left(\frac{u'}{u}-\frac{v'}{v}\right)\right\} &= 2uu'\left(\frac{u'}{u}-\frac{v'}{v}\right)+u^2\left(\frac{u''}{u}-\frac{v''}{v}\right)-u^2\left(\frac{u'^2}{u^2}-\frac{v'^2}{v^2}\right)\\ &= u\left(\frac{u'}{u}-\frac{v'}{v}\right)\left\{2u'-u\left(\frac{u'}{u}+\frac{v'}{v}\right)\right\}+u^2\{h(x)-g(x)\}\\ &= \frac{(u'v-uv')^2}{v^2}+u^2\{h(x)-g(x)\} > 0.\end{aligned}$$

Hence $u^2(u'/u-v'/v)$ is steadily increasing.

Suppose that u and v have the same number of zeros in (a,b).

The zero x_ν next before c is a zero of u and not of v, for between a and x_ν lie at least ν, and so exactly ν, zeros of v. Hence

$$u^2(b)\left\{\frac{u'(b)}{u(b)}-\frac{v'(b)}{v(b)}\right\} > u^2(x_\nu)\left\{\frac{u'(x_\nu)}{u(x_\nu)}-\frac{v'(x_\nu)}{v(x_\nu)}\right\} = 0, \qquad \frac{u'(b)}{u(b)} > \frac{v'(b)}{v(b)}.$$

It follows that $y'(b)/y(b)$ is a steadily decreasing function of λ in each interval (μ_m, μ_{m+1}); and it must decrease from ∞ to $-\infty$, since $y(b) = 0$ at each end, and $y'(b) \neq 0$. Hence there is just one value of λ in the interval such that

$$\frac{y'(b)}{y(b)} = -\cot\beta.$$

Hence *there is an increasing sequence of eigenvalues* $\lambda_0, \lambda_1, \ldots$ *such that the eigenfunction associated with* λ_m *has just* m *zeros in* $a < x < b$.

5.12. We now pass to the case in which the interval is $(0, \infty)$.

Let $q(x) \to \infty$ as $x \to \infty$. We shall show that this is quite similar to the Sturm-Liouville case; *there are discrete eigenvalues, and the eigenfunction associated with* λ_n *has* n *zeros.*

We consider the solution $y = y(x, \lambda)$ of the equation

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0$$

such that $y(0)$ and $y'(0)$ have given values. Since $q(x) \to \infty$, $q(x)-\lambda > 0$ for $x > x_\lambda$, say. Suppose that for some $x_1 > x_\lambda$

$$y(x_1, \lambda) > 0, \qquad y'(x_1, \lambda) > 0.$$

Then $y''(x, \lambda) > 0$ $(x \geqslant x_1)$, $y'(x, \lambda)$ increases steadily, and $y(x, \lambda) \to \infty$.

If
$$y(x_1, \lambda) > 0, \qquad y'(x_1, \lambda) < 0$$

there are two possibilities. If $y(x, \lambda)$ remains positive for $x \geqslant x_1$, then $y''(x, \lambda)$ remains positive, $y'(x, \lambda)$ increases steadily, and so tends to a limit (finite or infinite). This limit cannot be negative, or $y(x)$ would tend to $-\infty$; if it is positive, $y(x) \to \infty$. If $y'(x) \to 0$, then $y'(x, \lambda) < 0$ for $x \geqslant x_1$. Hence $y(x)$ is steadily decreasing, and so tends to a limit. Also

$$\int\limits_{x_1}^{x_2} \{q(x)-\lambda\}y(x)\, dx = \int\limits_{x_1}^{x_2} y''(x)\, dx$$
$$= y'(x_2)-y'(x_1) < -y'(x_1).$$

Hence $\{q(x)-\lambda\}y(x)$ is $L(x_1,\infty)$, and *a fortiori* $y(x)$ is $L(x_1,\infty)$. Hence $y(x) \to 0$. Hence also $y(x)$ is $L^2(x_1,\infty)$.

On the other hand, if $y(x)$ changes sign, it can do so at most once, and then tends to $-\infty$, by the previous argument.

Similar arguments hold if $y(x_1,\lambda) < 0$. Summing up, $y(x,\lambda)$ has at most one zero for $x > x_\lambda$, and either $y(x,\lambda) \to \pm\infty$, or $y(x,\lambda) \to 0$, $y'(x,\lambda) \to 0$, and $y(x,\lambda)$ is $L^2(x_1,\infty)$.

It follows that each $y(x,\lambda)$ has a finite number of zeros in $(0,\infty)$, say $n(\lambda)$. By the theorem of § 5.10, $n(\lambda)$ is a non-decreasing function of λ. The argument of § 5.11 shows that $n(\lambda) = 0$ if λ is negative and large enough, and that $n(\lambda) \to \infty$ as $\lambda \to \infty$.

Since $n(\lambda)$ changes by at least 1 at a discontinuity, it has only a finite number of discontinuities in any finite interval. Hence they are discrete points. We shall show that they are the eigenvalues of the differential equation.

In addition to the above results, we can show that, if $y(x,\lambda) \to 0$, and $\mu > \lambda$, then $y(x,\mu)$ has a zero greater than the greatest zero of $y(x,\lambda)$. Suppose e.g. that $y(x,\lambda) > 0$, $y'(x,\lambda) < 0$ for x large enough. Suppose that $y(x,\mu) > 0$ for $x \geqslant a$, where a is the greatest zero of $y(x,\lambda)$. If $y(x,\mu) \to 0$, (5.9.4) gives

$$(\mu-\lambda)\int\limits_a^\infty y(x,\lambda)y(x,\mu)\,dx = -y(a,\mu)y'(a,\lambda),$$

which gives a contradiction as before. Otherwise, if $y(x,\mu) > 0$ for large x, then $y'(x,\mu) > 0$ for large x. Hence

$$y(x,\mu)y'(x,\lambda)-y(x,\lambda)y'(x,\mu) < 0$$

for large x, and we obtain a contradiction again.

In particular it follows that to each interval where $n(\lambda) =$ constant corresponds at most one value of λ for which $y(x,\lambda)$ is L^2, and such a value must be the right-hand end-point of the interval.

Now consider a decreasing sequence λ_1, λ_2,... tending to a limit λ. Since $n(\lambda_p) = n(\lambda+0)$ for p large enough, we can suppose that

$$n(\lambda_1) = n(\lambda_2) = \ldots = k.$$

Let the zeros of $y(x,\lambda_p)$ be $a_1^{(p)},\ldots, a_k^{(p)}$. Then it follows from the above argument about the interlacing of zeros that for every $m \leqslant k$

$$a_m^{(1)} < a_m^{(2)} < \ldots < a_m^{(p)}.$$

Let c be a constant such that $q(x)-\lambda_1 \geqslant 0$ for $x \geqslant c$. Then $a_1^{(p)},\ldots, a_{k-1}^{(p)}$ remain less than c for all p, since $y(x, \lambda_p)$ has at most one zero greater than c. Hence

$$\lim_{p\to\infty} a_m^{(p)} = a_m \leqslant c$$

exists for $m = 1, 2,\ldots, k-1$, and the a_m are zeros of $y(x,\lambda)$. They are all different, since if two coincided we should have $y(a_m, \lambda) = 0$, $y'(a_m, \lambda) = 0$, which is impossible. Conversely, any zero of $y(x, \lambda)$ is a limit-point of the $a_m^{(p)}$. Hence $y(x, \lambda)$ has k or $k-1$ zeros according to whether $a_k^{(p)}$ tends to a finite limit or to infinity.

Now clearly $y'(a_k^{(p)}, \lambda_p)$ has the same sign for all p, say $+$. If $a_k^{(p)} \to \infty$, then $a_k^{(p)} > c$ for sufficiently large p. Since none of the $y(x, \lambda_p)$ are L^2, we must have

$$y'(x, \lambda_p) > 0, \qquad y(x, \lambda_p) < 0 \quad (c < x < a_k^{(p)}).$$

Making $p \to \infty$, it follows that

$$y'(x, \lambda) > 0, \qquad y(x, \lambda) < 0 \quad (x > c).$$

Hence $y(x, \lambda)$ is L^2.

On the other hand, if $a_k^{(p)}$ tends to a finite limit a_k, then

$$y'(x, \lambda) > 0, \qquad y(x, \lambda) > 0 \quad (x > a_k).$$

Hence $y(x, \lambda)$ is not L^2.

Summing up, either $n(\lambda) = n(\lambda+0)$ or $n(\lambda) = n(\lambda+0)-1$. In the latter case, $y(x, \lambda)$ is L^2, in the former case not.

If we argue similarly with an increasing sequence $\lambda_1, \lambda_2,\ldots,$ the $a_m^{(p)}$ are decreasing, and therefore all tend to finite limits. Hence $n(\lambda) = n(\lambda-0)$.

It follows that to every n corresponds an interval of values of λ in which $n(\lambda) = n$, and that the right-hand end-point of each such interval is such that $y(x, \lambda)$ is L^2.

Now let λ' and λ'' be interior points of an interval where $n(\lambda)$ is constant, and let x_0 be greater than the greatest zero of $y(x, \lambda)$ or $y'(x, \lambda)$ for $\lambda' \leqslant \lambda \leqslant \lambda''$. Then $y(x, \lambda)$ is positive increasing (or negative decreasing) for $x \geqslant x_0$, $\lambda' \leqslant \lambda \leqslant \lambda''$. Let m be the lower bound of $|y(x_0, \lambda)|$ for $\lambda' \leqslant \lambda \leqslant \lambda''$, so that $m > 0$. Then

$$|y(x, \lambda)| \geqslant m \quad (x \geqslant x_0,\ \lambda' \leqslant \lambda \leqslant \lambda'').$$

We can now take $y(x,\lambda)$ to be the function $\phi(x,\lambda)$ of Chapter III. By (3.4.1)

$$\chi(x,\lambda'')-\chi(x,\lambda') = \int_{\lambda'}^{\lambda''} \phi(x,u)\,dk(u)$$

$$\geqslant m\int_{\lambda'}^{\lambda''} dk(u)$$

$$\geqslant m\{k(\lambda''-0)-k(\lambda'+0)\}.$$

But $\chi(x,\lambda)$ is L^2 for every λ. Hence

$$k(\lambda'+0) = k(\lambda''-0).$$

Hence $k(\lambda)$ is constant in the open intervals where $n(\lambda)$ is constant. Let the points of discontinuity be λ_n, the saltus at λ_n, k_n. Then by (3.4.1)

$$\chi(x,\lambda) = \sum_{\kappa<\lambda_n<\lambda} k_n\,\phi(x,\lambda_n).$$

By (3.4.3)
$$g_1(\lambda) = \sum_{\kappa<\lambda_n<\lambda} k_n \int_0^\infty \phi(y,\lambda_n)f(y)\,dy$$

and by (3.6.3)

$$f(x) = \frac{1}{\pi}\sum_n k_n\,\phi(x,\lambda_n) \int_0^\infty \phi(y,\lambda_n)f(y)\,dy.$$

Hence the expansion is a series.

Also, comparing (3.5.1) with (3.4.4)–(3.4.6), we have

$$\rho(u_2)-\rho(-u_1) = \int_{u_1}^{u_2} \phi(x,u)\,dg_1(u).$$

Hence in the present case $\rho(x)$ is constant except at the points λ_n. Hence by (3.6.2) $\Phi(x,\lambda)$ is regular except for simple poles at the points λ_n.

It follows also that $m(\lambda)$ is meromorphic; for if $f(x) = 0$ for $x \geqslant X$, (2.3.2) gives

$$\Phi(x,\lambda) = \theta(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy + \phi(x,\lambda)\int_x^X \theta(y,\lambda)f(y)\,dy +$$
$$+m(\lambda)\phi(x,\lambda)\int_0^X \phi(y,\lambda)f(y)\,dy,$$

and the result follows.

5.13. The interval $(-\infty, \infty)$. Theorems on the nature of the spectrum when the interval is $-\infty < x < \infty$ can be deduced from those already obtained for the interval $(0, \infty)$.

Let $q(x) \to \infty$ as $x \to -\infty$ and as $x \to \infty$. Then both $m_1(\lambda)$ and $m_2(\lambda)$ are meromorphic. Hence the functions

$$\frac{1}{m_1(\lambda)-m_2(\lambda)}, \quad \frac{m_1(\lambda)}{m_1(\lambda)-m_2(\lambda)}, \quad \frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda)-m_2(\lambda)} \tag{5.13.1}$$

occurring in the $(-\infty, \infty)$ formula are also meromorphic, so that the corresponding expansion is a series and the spectrum is discrete. The argument of § 5.12 also extends without difficulty to this case.

Next let $q(x) \to \infty$ as $x \to \infty$, and $q(x) \to -\infty$ as $x \to -\infty$, $\int\limits_{-\infty} |q(x)|^{-\frac{1}{2}}\, dx$ being divergent. Then $\mathbf{I}\{m_1(\lambda)\}$ tends to a finite non-zero limit along the whole real axis, while $\mathbf{I}\{m_2(\lambda)\}$ tends to zero in general but to infinity at certain discrete points. Hence the imaginary parts of the functions (5.13.1) tend to finite limits, which can vanish at most at discrete points. The spectrum therefore extends continuously from $-\infty$ to ∞. In fact it is clear that, if $q(x) \to \infty$ at one end of the interval $(-\infty, \infty)$, the result is the same as if we had the other half-interval only with a given boundary condition at $x = 0$.

REFERENCES

§§ 5.2–5.3. Weyl (2), Titchmarsh (7).
§ 5.4. This transformation occurs in Langer (4), § 3.
§§ 5.7–5.8. So far as I know this is new.
§§ 5.9–5.11. Classical.
§ 5.12. Weyl (2).

VI

A SPECIAL CONVERGENCE THEOREM

6.1. In Chapter III it was proved that the general expansion formula represents an 'arbitrary' function $f(x)$, provided that $f(x)$ is twice differentiable, that $f(x)$ and $f''(x)-q(x)f(x)$ are $L^2(0,\infty)$, and that the 'boundary condition at infinity' (2.7.2) is satisfied. In this and later chapters we shall show that, by imposing further conditions on $q(x)$, we can relax those bearing on $f(x)$; and in fact it will be shown that in certain cases the expansion converges to $f(x)$ if $f(x)$ satisfies conditions similar to those for the convergence of an ordinary Fourier series.

In this chapter it will be assumed that the integral

$$\int_0^\infty |q(x)|\,dx \tag{6.1.1}$$

is convergent. This is perhaps not the most interesting case, but it is the simplest, and the analysis suggests the method which will be used later in considering cases in which $q(x)\to\infty$. The advantage of this and other special assumptions is that they enable us to prove asymptotic formulae for the functions $\phi(x,\lambda)$ and $\psi(x,\lambda)$. In the general case these formulae are not known, and we have to rely on a direct discussion of the function $\Phi(x,\lambda)$.

6.2. What we require primarily is a solution of the equation

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y=0 \tag{6.2.1}$$

which is small when $\mathbf{I}(\lambda)$ is large and positive. Consider the integral equation

$$\chi(x)=e^{isx}+\frac{1}{2is}\int_0^x e^{is(x-y)}q(y)\chi(y)\,dy+\frac{1}{2is}\int_x^\infty e^{is(y-x)}q(y)\chi(y)\,dy, \tag{6.2.2}$$

in which $\lambda=s^2$. On differentiating this twice, it is at once verified (formally) that $y=\chi(x)$ satisfies (6.2.1). Now a solution of (6.2.2) can be obtained as follows.

Let
$$\chi_1(x)=e^{isx} \tag{6.2.3}$$

and, for $n \geqslant 1$,

$$\chi_{n+1}(x) = e^{isx}+\frac{1}{2is}\int\limits_0^x e^{is(x-y)}q(y)\chi_n(y)\,dy+$$

$$+\frac{1}{2is}\int\limits_x^\infty e^{is(y-x)}q(y)\chi_n(y)\,dy. \tag{6.2.4}$$

Then

$$\chi_2(x)-\chi_1(x) = \frac{e^{isx}}{2is}\left\{\int\limits_0^x q(y)\,dy+\int\limits_x^\infty e^{2is(y-x)}q(y)\,dy\right\}.$$

Let $s = \sigma+it$ $(t \geqslant 0)$, and let J denote the value of the integral (6.1.1). Then

$$|\chi_2(x)-\chi_1(x)| \leqslant \frac{e^{-tx}J}{2|s|}. \tag{6.2.5}$$

Next

$$\chi_3(x)-\chi_2(x) = \frac{1}{2is}\int\limits_0^x e^{is(x-y)}q(y)\{\chi_2(y)-\chi_1(y)\}\,dy+$$

$$+\frac{1}{2is}\int\limits_x^\infty e^{is(y-x)}q(y)\{\chi_2(y)-\chi_1(y)\}\,dy,$$

whence by (6.2.3)

$$|\chi_3(x)-\chi_2(x)| \leqslant \frac{e^{-tx}J}{|2s|^2}\left\{\int\limits_0^x |q(y)|\,dy+\int\limits_x^\infty e^{2t(x-y)}|q(y)|\,dy\right\}$$

$$\leqslant \left(\frac{J}{|2s|}\right)^2 e^{-tx}; \tag{6.2.6}$$

and so on generally. It follows that, if $|s| > \frac{1}{2}J$, the series

$$\sum_{n=1}^\infty \{\chi_{n+1}(x)-\chi_n(x)\}$$

is convergent, i.e. that $\chi_n(x)$ tends to a limit, say $\chi(x)$.

For every n

$$|\chi_n(x)| \leqslant |\chi_1(x)|+|\chi_2(x)-\chi_1(x)|+...+|\chi_n(x)-\chi_{n-1}(x)|$$

$$\leqslant e^{-tx}\left(1+\frac{J}{2|s|}+...\right)$$

$$= \frac{e^{-tx}}{1-\frac{1}{2}J/|s|}, \tag{6.2.7}$$

and $\chi(x)$ satisfies the same inequality. It follows by 'dominated convergence' that we can make $n\to\infty$ under the integral sign in (6.2.4). Hence $\chi(x)$ satisfies (6.2.2), and so also (6.2.1).

For a fixed s, or s in a bounded part of the region $t \geqslant 0$, $|s| > \frac{1}{2}J$, (6.2.2) now gives

$$\chi(x) = e^{isx} + \frac{e^{isx}}{2is} \int_0^\infty e^{-isy} q(y)\chi(y)\,dy -$$
$$- \frac{e^{isx}}{2is} \int_x^\infty e^{-isy} q(y)\chi(y)\,dy + \frac{e^{isx}}{2is} \int_x^\infty e^{is(y-2x)} q(y)\chi(y)\,dy.$$

The last two integrals tend to 0 as $x \to \infty$; hence

$$\chi(x) = e^{isx}\{K(\lambda)+o(1)\}, \tag{6.2.8}$$

where
$$K(\lambda) = 1 + \frac{1}{2is} \int_0^\infty e^{-isy} q(y)\chi(y)\,dy. \tag{6.2.9}$$

6.3. Let $\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$ denote as usual the solution belonging to $L^2(0,\infty)$. Then

$$\psi(x,\lambda) = K_1(\lambda)\chi(x,\lambda)+K_2(\lambda)\phi(x,\lambda).$$

Now $\chi(x,\lambda)$ is $L^2(0,\infty)$, but $\phi(x,\lambda)$ is not, by (5.3.5), at any rate if $|\lambda|$ is large enough; for (5.3.6) gives

$$M(\lambda) \sim \tfrac{1}{2}\sin\alpha \quad (\sin\alpha \neq 0),$$
$$\sim \frac{\cos\alpha}{2is} \quad (\sin\alpha = 0).$$

Hence $K_2(\lambda) = 0$, i.e.

$$\psi(x,\lambda) = K_1(\lambda)\chi(x,\lambda).$$

As in § 5.3, the asymptotic formulae (5.3.5) and (6.2.8) can be differentiated, i.e. we can apply similar arguments to the differentiated functions, and obtain (as $x \to \infty$)

$$\phi'(x) \sim -isM(\lambda)e^{-isx},$$
$$\chi'(x) \sim isK(\lambda)e^{isx}.$$

Hence

$$W(\phi,\chi) \sim M(\lambda)isK(\lambda)+isM(\lambda)K(\lambda) = 2isK(\lambda)M(\lambda).$$

Hence
$$W(\phi,\psi) \sim 2isK(\lambda)M(\lambda)K_1(\lambda).$$

But $W(\phi,\psi) = 1$. Hence

$$2isK(\lambda)M(\lambda)K_1(\lambda) = 1.$$

Hence
$$\psi(x,\lambda) = \frac{\chi(x,\lambda)}{2isK(\lambda)M(\lambda)}.$$

6.4. THEOREM 6.4. *Let $q(x)$ be $L(0,\infty)$, and $f(x)$ $L^2(0,\infty)$ and also $L(0,\infty)$. Then the eigenfunction expansion is*

$$\frac{1}{\pi}\int\limits_0^\infty \frac{\phi(x,\lambda)}{\lambda^{\frac{1}{2}}\{\mu^2(\lambda)+\nu^2(\lambda)\}}\,d\lambda \int\limits_0^\infty \phi(y,\lambda)f(y)\,dy+$$

$$+\sum r_n\,\phi(x,\lambda_n)\int\limits_0^\infty \phi(y,\lambda_n)f(y)\,dy,$$

where $\mu(\lambda)$ and $\nu(\lambda)$ are defined in § 5.3, and, in the sum, r_n $(n = 1, 2,...)$ runs through the residues of $m(\lambda)$ at its poles on the negative real axis, and πr_0 is the saltus of $k(\lambda)$ at $\lambda = 0$. The expansion converges under the same conditions as an ordinary Fourier series; e.g. if f is of bounded variation in the neighbourhood of x, its value is

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Let

$$\begin{aligned}\Phi(x,\lambda) &= \psi(x,\lambda)\int\limits_0^x \phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int\limits_0^\infty \psi(y,\lambda)f(y)\,dy\\ &= \psi(x,\lambda)\left(\int\limits_0^{x-\delta}+\int\limits_{x-\delta}^{x}\right)+\phi(x,\lambda)\left(\int\limits_x^{x+\delta}+\int\limits_{x+\delta}^{\infty}\right)\\ &= \Phi_1+\Phi_2+\Phi_3+\Phi_4.\end{aligned}$$

Consider
$$\int\limits_{-R+i\epsilon}^{R+i\epsilon} \Phi(x,\lambda)\,d\lambda.$$

This has the same value whether it is taken along the straight line, or round the semicircle above it. Suppose, for example, that $\sin\alpha \neq 0$ in the above formulae. Then for $|s| \geqslant J$

$$|\psi(y,\lambda)| \leqslant \frac{2e^{-ty}}{2|s|\,|M(\lambda)K(\lambda)|} < \frac{Ae^{-ty}}{|s|}.$$

Also
$$\phi(x,\lambda) = O(e^{tx}).$$

Hence
$$\Phi_4(x,\lambda) = O\left\{\frac{e^{tx}}{|s|}\int\limits_{x+\delta}^{\infty} e^{-ty}|f(y)|\,dy\right\} = O\left(\frac{e^{-t\delta}}{|s|}\right).$$

The integral of this round the semicircle tends to 0 as $R\to\infty$, for any positive δ. A similar argument clearly applies to Φ_1.

Now consider Φ_3. For x fixed, or in a finite interval,

$$|\chi(x)-e^{isx}| \leqslant e^{-tx}\left(\frac{J}{2|s|}+...\right) < \frac{Ae^{-tx}}{|s|}.$$

Also $\quad K(\lambda) = 1+O(|s|^{-1}), \qquad M(\lambda) = \frac{1}{2}\sin\alpha+O(|s|^{-1}).$

Hence
$$\psi(x,\lambda) = \frac{e^{isx}}{is\sin\alpha}\left\{1+O\left(\frac{1}{|s|}\right)\right\}.$$

Also
$$\phi(x,\lambda) = \cos sx\sin\alpha\left(1+O\frac{1}{|s|}\right)$$

by (1.7.3). Hence
$$\Phi_3(x,\lambda) = \frac{e^{isx}+e^{-isx}}{2is}\int\limits_x^{x+\delta} e^{isy}f(y)\,dy+O\left\{\frac{e^{tx}}{|s|^2}\int\limits_x^{x+\delta} e^{-ty}f(y)\,dy\right\}.$$

The last term is
$$O\left\{\frac{1}{|\lambda|}\int\limits_x^{x+\delta} |f(y)|\,dy\right\},$$

and the integral of this round the semicircle is
$$O\left\{\int\limits_x^{x+\delta} |f(y)|\,dy\right\},$$

which can be made as small as we please, by choice of δ. The term involving e^{isx} also gives a zero limit. The other term is the same as in the case of an ordinary Fourier series; and similarly for Φ_2. Altogether it follows that, e.g. in the bounded variation case,
$$\lim_{R\to\infty}\int\limits_{-R+i\epsilon}^{R+i\epsilon} \Phi(x,\lambda)\,d\lambda \to \tfrac{1}{2}i\pi\{f(x+0)+f(x-0)\}.$$

This is true uniformly for $0<\epsilon\leqslant 1$.

Since $f(x)$ is $L^2(0,\infty)$, the analysis of § 3.4 applies, so that
$$\lim_{\to 0}\mathbf{I}\left\{-\frac{1}{\pi}\int\limits_{-R+i\epsilon}^{R+i\epsilon}\Phi(x,\lambda)\,d\lambda\right\} = \frac{1}{\pi}\int\limits_{-R}^{R}\phi(x,\lambda)\,dg_1(\lambda),$$

where
$$g_1(\lambda) = \int\limits_0^\infty \chi(y,\lambda)f(y)\,dy, \qquad \chi(y,\lambda) = \int\limits_\kappa^\lambda \phi(y,u)\,dk(u).$$

By (5.3.10)
$$\chi(y,\lambda) = \int\limits_\kappa^\lambda \frac{\phi(y,u)}{u^{\frac{1}{2}}\{\mu^2(u)+\nu^2(u)\}}\,du,$$

taking $\kappa>0$. Since $\phi(y,u)$ is bounded for $0<\kappa\leqslant u\leqslant\lambda$, $y\to\infty$, it follows that
$$g_1(\lambda) = \int\limits_\kappa^\lambda \frac{du}{u^{\frac{1}{2}}\{\mu^2(u)+\nu^2(u)\}}\int\limits_0^\infty \phi(y,u)f(y)\,dy.$$

Hence, if $\rho > 0$,

$$\int_{\rho}^{R} \phi(x,\lambda)\, dg_1(\lambda) = \int_{\rho}^{R} \frac{\phi(x,\lambda)\, d\lambda}{\lambda^{\frac{1}{2}}\{\mu^2(\lambda)+\nu^2(\lambda)\}} \int_{0}^{\infty} \phi(y,\lambda)f(y)\, dy.$$

In the interval $-R < \lambda < -\rho'$ $(\rho' > 0)$, $g_1(\lambda)$ is constant except for a finite number of discontinuities at the poles of $m(\lambda)$. Hence

$$\int_{-R}^{-\rho'} \phi(x,\lambda)\, dg_1(\lambda) = \pi \sum_{-R<\lambda_n<-\rho'} r_n \phi(x,\lambda_n) \int_{0}^{\infty} \phi(y,\lambda_n)f(y)\, dy,$$

each $\phi(y,\lambda_n)$ being $L^2(0,\infty)$, as in § 2.5. Finally

$$\lim_{\rho,\rho'\to 0} \int_{-\rho'}^{\rho} \phi(x,\lambda)\, dg_1(\lambda) = R_0\phi(x,0),$$

where R_0 is the saltus of $g_1(\lambda)$ at $\lambda = 0$. It easily follows from the analysis of § 3.4 that

$$R_0 = \pi r_0 \int_{0}^{\infty} \phi(y,0)f(y)\, dy,$$

where r_0 is defined above. This completes the proof.

The form of the result makes it seem possible that the condition that $f(x)$ is L^2 may be unnecessary; it has been introduced here so that previous theorems which have been proved with this condition may be used.

REFERENCE

Stone (1).

VII

THE DISTRIBUTION OF THE EIGENVALUES

7.1. In this chapter we suppose that $q(x) \to \infty$, so that the differential equation

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0 \tag{7.1.1}$$

has discrete eigenvalues $\lambda_0, \lambda_1, \ldots$. We then ask how the distribution of these eigenvalues is determined by the function $q(x)$.

If $\lambda = \lambda_n$, then $y = \psi_n(x)$ has just n real zeros (§ 5.12). On the other hand, by comparing the distribution of the zeros of (7.1.1) with that of the zeros of known functions, we can obtain approximate formulae for the number of zeros in terms of λ_n. Approximate relations between n and λ_n are thus obtained.

To carry out the analysis considerable restrictions have to be imposed on the function $q(x)$. We begin by assuming that $q'(x)$ and $q''(x)$ exist, and that $q'(x) > 0$, $q''(x) \geqslant 0$ for $x > 0$. It is also convenient to assume that $q(0) = 0$. This is no restriction, since it merely involves a choice of the λ-origin.

Let $p = p(\lambda)$ be defined by the equation $q(p) = \lambda$, and let

$$\xi = \xi(\lambda,x) = \int\limits_0^x \{\lambda-q(t)\}^{\frac{1}{2}}\,dt \quad (0 \leqslant x \leqslant p). \tag{7.1.2}$$

Let $\phi(\xi)$ denote (temporarily) a twice-differentiable function of ξ, and let

$$Y = Y(x) = \{\lambda-q(x)\}^{-\frac{1}{4}}\phi(\xi). \tag{7.1.3}$$

Then
$$\frac{Y'}{Y} = \frac{\phi'(\xi)}{\phi(\xi)}\xi'+\frac{1}{4}\frac{q'(x)}{\lambda-q(x)},$$

$$\frac{Y''}{Y}-\frac{Y'^2}{Y^2} = \left\{\frac{\phi''(\xi)}{\phi(\xi)}-\frac{\phi'^2(\xi)}{\phi^2(\xi)}\right\}\xi'^2+\frac{\phi'(\xi)}{\phi(\xi)}\xi''+\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}+\frac{1}{4}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}.$$

Hence

$$\begin{aligned}\frac{Y''}{Y} &= \frac{\phi''(\xi)}{\phi(\xi)}\xi'^2+\frac{\phi'(\xi)}{\phi(\xi)}\left\{\xi''+\frac{\frac{1}{2}\xi'q'(x)}{\lambda-q(x)}\right\}+\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}+\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2} \\ &= \{\lambda-q(x)\}\frac{\phi''(\xi)}{\phi(\xi)}+\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}+\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}.\end{aligned} \tag{7.1.4}$$

In particular, if $\phi(\xi) = \cos\xi$, then $\phi''(\xi) = -\phi(\xi)$, and Y satisfies the differential equation

$$\frac{d^2Y}{dx^2}+\left\{\lambda-q(x)-\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}-\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}\right\}Y = 0. \tag{7.1.5}$$

7.2. THEOREM 7.2. *If $q(x)$ satisfies the above conditions,*

$$n > \frac{1}{\pi}\int\limits_0^{p_n} \{\lambda_n-q(x)\}^{\frac{1}{2}}\,dx - \tfrac{3}{2}, \tag{7.2.1}$$

where $p_n = p(\lambda_n)$.

For $0 < x < p$, the coefficient of y in (7.1.1) is greater than that of Y in (7.1.5). Then by the theorem of § 5.9, between any two zeros of Y there is at least one zero of y. Hence if Y has l zeros in $(0, p_n)$, then $n \geqslant l-1$.

The zeros of Y, i.e. of $\cos\xi$, are at the points $\xi = (m+\frac{1}{2})\pi$, for values of m for which this lies between 0 and

$$\int\limits_0^{p_n} \{\lambda_n-q(t)\}^{\frac{1}{2}}\,dt.$$

Hence
$$(l-\tfrac{1}{2})\pi \leqslant \int\limits_0^{p_n} \{\lambda_n-q(t)\}^{\frac{1}{2}}\,dt < (l+\tfrac{1}{2})\pi,$$

and the theorem follows.

Since $q(x)$ is convex downwards, $q(\frac{1}{2}p_n) \leqslant \frac{1}{2}\lambda_n$, and

$$n > \frac{1}{\pi}\int\limits_0^{\frac{1}{2}p_n} \{\lambda_n-q(x)\}^{\frac{1}{2}}\,dx - \tfrac{3}{2}$$

$$> \frac{1}{2\pi}p_n(\tfrac{1}{2}\lambda_n)^{\frac{1}{2}} - \tfrac{3}{2}.$$

Hence
$$\lambda_n < 8\pi^2(n+\tfrac{3}{2})^2/p_n^2. \tag{7.2.2}$$

In particular, $\lambda_n = o(n^2)$.

Actually we can prove this if $q(x)$ is any function which is twice differentiable and tends to infinity; for compare the solution of (7.1.1) with

$$Y_1 = \{\lambda-1-q(x)\}^{-\frac{1}{4}}\cos\left[\int\limits_0^{x} \{\lambda-1-q(t)\}^{\frac{1}{2}}\,dt\right].$$

The coefficient of y in (7.1.1) exceeds that of Y_1, in the differential

equation for Y_1, over any finite interval $(0, X)$, if X is fixed and λ is large enough. Hence

$$n > \frac{1}{\pi}\int_0^X \{\lambda_n - 1 - q(x)\}^{\frac{1}{2}}\, dx - \tfrac{3}{2}$$
$$> AX\lambda_n^{\frac{1}{2}}.$$

Since X may be arbitrarily large, the result follows.

In the Sturm-Liouville (finite interval) case we have of course $\lambda_n \sim An^2$.

In the case in which the interval is $(-\infty, \infty)$, and $q(x)$ is an even function, and $q'(0) = 0$, there is a slightly more precise result, which will be of use later. In this case, if n is even, $\psi_n(x)$ is an even function, with $\frac{1}{2}n$ positive zeros and $\frac{1}{2}n$ negative zeros. Let

$$Y(x) = \psi_n(0)\lambda_n^{\frac{1}{4}}\{\lambda_n - q(x)\}^{-\frac{1}{4}}\cos\left[\int_0^x \{\lambda_n - q(t)\}^{\frac{1}{2}}\, dt\right].$$

Then $Y(0) = \psi_n(0)$, $Y'(0) = 0 = \psi_n'(0)$. Hence, by the theorem of § 5.10, $\psi_n(x)$ has at least as many zeros in $(0, p_n)$ as $Y(x)$ has. If $Y(x)$ has l zeros in this interval, then

$$\int_0^{p_n} \{\lambda_n - q(t)\}^{\frac{1}{2}}\, dt \leqslant (l+\tfrac{1}{2})\pi \leqslant (\tfrac{1}{2}n+\tfrac{1}{2})\pi.$$

Hence
$$n \geqslant \frac{2}{\pi}\int_0^{p_n} \{\lambda_n - q(t)\}^{\frac{1}{2}}\, dt - 1. \tag{7.2.3}$$

If n is odd, $\psi_n(x)$ has $\frac{1}{2}n - \frac{1}{2}$ positive zeros. Let

$$Y(x) = \psi_n'(0)\lambda_n^{-\frac{1}{4}}\{\lambda_n - q(x)\}^{-\frac{1}{4}}\sin\left[\int_0^x \{\lambda_n - q(t)\}^{\frac{1}{2}}\, dt\right].$$

Then $Y(0) = 0 = \psi_n(0)$, $Y'(0) = \psi_n'(0)$. Hence, if $Y(x)$ has l zeros for $0 < x \leqslant p_n$,

$$\int_0^{p_n} \{\lambda_n - q(t)\}^{\frac{1}{2}}\, dt < (l+1)\pi \leqslant (\tfrac{1}{2}n+\tfrac{1}{2})\pi,$$

and (7.2.3) follows again.

7.3. In the problem of the upper bound of n, we have to assume rather more about $q(x)$.

THEOREM 7.3. *Let $q(x)$ satisfy the above conditions, and also*

$$q''(x) = o[\{q'(x)\}^{\frac{4}{3}}]. \tag{7.3.1}$$

Then $$n \sim \frac{1}{\pi}\int_0^{p_n} \{\lambda_n - q(x)\}^{\frac{1}{2}}\,dx. \tag{7.3.2}$$

We observe that, if $q'(x)$ is non-decreasing and

$$q''(x) = O[\{q'(x)\}^{\gamma}] \quad (1 < \gamma < 2), \tag{7.3.3}$$

then

$$\begin{aligned} q'(x) &= \int_0^x q''(t)\,dt + q'(0) \\ &= O\left[\int_0^x \{q'(t)\}^{\gamma}\,dt\right] \\ &= O\left[\{q'(x)\}^{\gamma-1}\int_0^x q'(t)\,dt\right] \\ &= O[\{q'(x)\}^{\gamma-1}q(x)], \end{aligned}$$

$$\{q'(x)\}^{2-\gamma} = O\{q(x)\}, \qquad q'(x) = O\{q(x)\}^{1/(2-\gamma)}; \tag{7.3.4}$$

and similarly with o instead of O. Thus (7.3.1) gives

$$q'(x) = o[\{q(x)\}^{\frac{3}{2}}]. \tag{7.3.5}$$

All such conditions imply a certain restriction on the regularity of the growth of $q(x)$, but not on its rate of growth.

Now let Y_1 be the function obtained by replacing λ by $\lambda+\mu$ ($\mu > 0$) in Y, with $\phi(\xi) = \cos\xi$. Then the coefficient of Y_1, in the differential equation for Y_1 corresponding to (7.1.5), exceeds $\lambda - q(x)$ if

$$\frac{5q'^2(x)}{16\{\lambda+\mu-q(x)\}^2} + \frac{q''(x)}{4\{\lambda+\mu-q(x)\}} < \mu. \tag{7.3.6}$$

This is true for $0 \leqslant x \leqslant p_n$ if

$$\frac{5q'^2(p_n)}{16\mu^2} + \frac{q''(p_n)}{4\mu} < \mu.$$

Let $\mu = \{q'(p_n)\}^{\frac{2}{3}}$. Using (7.3.1), this is seen to be true if p_n is large enough.

If the number of zeros of Y_1 in the interval $(0, p_n)$ is m, then $m \geqslant n-1$, and

$$(m-\tfrac{1}{2})\pi \leqslant \int_0^{p_n} \{\lambda_n+\mu-q(x)\}^{\frac{1}{2}}\,dx < (m+\tfrac{1}{2})\pi.$$

Hence $$n \leqslant \frac{1}{\pi}\int_0^{p_n} \{\lambda_n+\mu-q(x)\}^{\frac{1}{2}}\,dx + \tfrac{3}{2}.$$

To prove (7.3.2) we now require

$$\int_0^{p_n} [\{\lambda_n+\mu-q(x)\}^{\frac{1}{2}}-\{\lambda_n-q(x)\}^{\frac{1}{2}}]\,dx = o\left[\int_0^{p_n} \{\lambda_n-q(x)\}^{\frac{1}{2}}\,dx\right]. \tag{7.3.7}$$

Now

$$\{\lambda_n+\mu-q(x)\}^{\frac{1}{2}}-\{\lambda_n-q(x)\}^{\frac{1}{2}} = \frac{\mu}{\{\lambda_n+\mu-q(x)\}^{\frac{1}{2}}+\{\lambda_n-q(x)\}^{\frac{1}{2}}} \leqslant \mu^{\frac{1}{2}}$$

for $q(x) \leqslant \lambda_n$. Also, since $q(x)$ is convex downwards, $q(\frac{1}{2}p_n) \leqslant \frac{1}{2}\lambda_n$. Hence

$$\int_0^{p_n} \{\lambda_n-q(x)\}^{\frac{1}{2}}\,dx \geqslant \int_0^{\frac{1}{2}p_n} (\tfrac{1}{2}\lambda_n)^{\frac{1}{2}}\,dx = \tfrac{1}{2}p_n(\tfrac{1}{2}\lambda_n)^{\frac{1}{2}}.$$

Hence (7.3.7) is true if $\mu = o(\lambda_n)$, which is true by (7.3.4). This proves the theorem.

Suppose, for example, that $q(x) = x^k$ $(k > 0)$. Then $p_n = \lambda_n^{1/k}$, and

$$\int_0^{p_n} \{\lambda_n-q(x)\}^{\frac{1}{2}}\,dx = \int_0^{\lambda_n^{1/k}} (\lambda_n-x^k)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{k}\lambda_n^{\frac{1}{2}+1/k}\int_0^1 (1-t)^{\frac{1}{2}}t^{1/k-1}\,dt$$

$$= \frac{\Gamma(\frac{3}{2})\Gamma(1/k)}{k\Gamma(\frac{3}{2}+1/k)}\lambda_n^{\frac{1}{2}+1/k}.$$

Hence it follows from the theorem that

$$\lambda_n \sim \left\{\frac{\pi k\Gamma(\frac{3}{2}+1/k)}{\Gamma(\frac{3}{2})\Gamma(1/k)}n\right\}^{2k/(k+2)}. \tag{7.3.8}$$

This is easily verified in particular cases; e.g. if $k = 1$, the formulae are those of § 4.12. As $\lambda \to \infty$

$$J_{\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}})+J_{-\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}}) \sim \left(\frac{3}{\pi\lambda^{\frac{3}{2}}}\right)^{\frac{1}{2}}\{\cos(\tfrac{2}{3}\lambda^{\frac{3}{2}}-\tfrac{1}{6}\pi-\tfrac{1}{4}\pi)+\cos(\tfrac{2}{3}\lambda^{\frac{3}{2}}+\tfrac{1}{6}\pi-\tfrac{1}{4}\pi)\}$$

$$= \frac{3}{\pi^{\frac{1}{2}}\lambda^{\frac{3}{4}}}\cos(\tfrac{2}{3}\lambda^{\frac{3}{2}}-\tfrac{1}{4}\pi).$$

Hence

$$\tfrac{2}{3}\lambda_n^{\frac{3}{2}}-\tfrac{1}{4}\pi \sim (n+\tfrac{1}{2})\pi,$$

agreeing with (7.3.8).

7.4. The following well-known property of Bessel functions is now required.

LEMMA 7.4. *In the interval $(0, X)$, $J_\nu(x)$ has $X/\pi+O(1)$ zeros.*

See Watson, *Theory of Bessel Functions*, § 15.4; alternatively, consider

$$f(x) = \frac{J_\nu(x)}{x^\nu} = \left\{\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+O\left(\frac{1}{x^{\frac{3}{2}}}\right)\right\}\frac{1}{x^\nu}.$$

It is clear from this formula that $f(x)$ has at least one zero in each interval

$$m\pi+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi-\delta \leqslant x \leqslant m\pi+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+\delta$$

if δ is a fixed positive number less than $\frac{1}{2}\pi$, and m is large enough; for $f(x)$ has opposite signs at the ends of such an interval. Also

$$f'(x) = -\frac{J_{\nu+1}(x)}{x^\nu} = -\left\{\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos(x-\tfrac{1}{2}\nu\pi-\tfrac{3}{4}\pi)+O\left(\frac{1}{x^{\frac{3}{2}}}\right)\right\}\frac{1}{x^\nu}.$$

Hence $f'(x)$ is of constant sign throughout each such interval, so that $f(x)$ has at most one zero in the interval. The result now follows.

7.5. Application of Bessel functions. In the problem of the upper bound of n, the comparison function used above is not the best that can be found. It was observed by Langer that a better approximation to a solution of (7.1.1) is obtained by using Bessel functions.

Let us now take

$$\phi(\xi) = \{\xi(p)-\xi\}^{\frac{1}{2}}J_\nu\{\xi(p)-\xi\}. \tag{7.5.1}$$

By (1.11.2)

$$\frac{\phi''(\xi)}{\phi(\xi)} = \frac{\nu^2-\frac{1}{4}}{\{\xi(p)-\xi(x)\}^2}-1.$$

Hence (7.1.4) gives

$$\frac{d^2Y}{dx^2}+\left[\lambda-q(x)-\frac{(\nu^2-\frac{1}{4})\{\lambda-q(x)\}}{\{\xi(p)-\xi(x)\}^2}-\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}-\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}\right]Y = 0. \tag{7.5.2}$$

If $\nu = \pm\frac{1}{2}$, we return to the previous formulae. If $\nu = \frac{1}{3}$, the infinity of the new term at $x = p$ just cancels those of the following terms. It is clear that this must be so; for as $\xi \to \xi(p)$

$$\{\xi(p)-\xi\}^{\frac{1}{2}}J_{\frac{1}{3}}\{\xi(p)-\xi\} \sim A\{\xi(p)-\xi\}^{\frac{5}{6}}$$

and

$$\xi(p)-\xi = \int\limits_x^p \{\lambda-q(t)\}^{\frac{1}{2}}\,dt \sim \frac{1}{q'(x)}\int\limits_x^p \{\lambda-q(t)\}^{\frac{1}{2}}\,dt$$

$$= \tfrac{2}{3}\{\lambda-q(x)\}^{\frac{3}{2}}/q'(x).$$

Hence

$$Y \sim A\{q'(x)\}^{-\frac{5}{6}}\{\lambda-q(x)\}$$

and no negative powers of $\lambda-q(x)$ arise on differentiating.

For our purpose, however, it is better to take $\nu = 0$, so as to have something to spare in comparing (7.1.1) with (7.5.2).

Let $N(x)+1$ be the number of eigenvalues not exceeding x, so that

$$N(\lambda) = n \quad (\lambda_{n-1} < \lambda \leqslant \lambda_n).$$

Let $x = p(y)$ be the inverse function of $y = q(x)$, so that $p(\lambda) = p$, $p(\lambda_n) = p_n$. Then

$$N(\lambda) \geqslant \frac{1}{\pi}\int\limits_0^p \{\lambda-q(x)\}^{\frac{1}{2}}\,dx + O(1). \tag{7.5.3}$$

For this is true for each λ_n, by Theorem 7.2; and so, if $\lambda_{n-1} < \lambda \leqslant \lambda_n$ (denoting the integral by $I(\lambda)$),

$$\begin{aligned} N(\lambda) &\geqslant N(\lambda_n)-1 \\ &\geqslant \frac{1}{\pi}I(\lambda_n)+O(1) \\ &\geqslant \frac{1}{\pi}I(\lambda)+O(1), \end{aligned}$$

since $I(\lambda)$ is steadily increasing.

We shall now prove

THEOREM 7.5. *Let $q'(x) \to \infty$, $q''(x) \geqslant 0$, and*

$$q''(x) \leqslant \{q'(x)\}^{\gamma} \quad (1 < \gamma < \tfrac{4}{3}) \tag{7.5.4}$$

for $x > x_0$. Then

$$N(\lambda) = \frac{1}{\pi}\int\limits_0^p \{\lambda-q(x)\}^{\frac{1}{2}}\,dx + O(1). \tag{7.5.5}$$

Let

$$\chi(x,\lambda) = \frac{\lambda-q(x)}{4\{\xi(p)-\xi(x)\}^2} - \frac{1}{4}\frac{q''(x)}{\lambda-q(x)} - \frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}, \tag{7.5.6}$$

so that (7.5.2), with $\nu = 0$, is

$$\frac{d^2Y}{dx^2}+\{\lambda-q(x)+\chi(x,\lambda)\}Y = 0. \tag{7.5.7}$$

Now

$$\begin{aligned} \xi(p)-\xi(x) &= \int\limits_x^p \{\lambda-q(t)\}^{\frac{1}{2}}q'(t)\frac{1}{q'(t)}\,dt \\ &\leqslant \frac{1}{q'(x)}\int\limits_x^p \{\lambda-q(t)\}^{\frac{1}{2}}q'(t)\,dt \\ &= \tfrac{2}{3}\{\lambda-q(x)\}^{\frac{3}{2}}/q'(x). \end{aligned} \tag{7.5.8}$$

Hence

$$\chi(x,\lambda) \geqslant \frac{9}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}-\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}-\frac{5}{16}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}$$

$$= \frac{1}{4}\frac{q'^2(x)}{\{\lambda-q(x)\}^2}-\frac{1}{4}\frac{q''(x)}{\lambda-q(x)}. \tag{7.5.9}$$

Hence $\chi(x,\lambda) \geqslant 0$ provided that

$$\lambda-q(x) \leqslant q'^2(x)/q''(x), \tag{7.5.10}$$

or, by (7.5.4), provided that

$$\lambda-q(x) \leqslant \{q'(x)\}^{2-\gamma}. \tag{7.5.11}$$

If (7.5.11) holds for $x_1 \leqslant x \leqslant p$, it follows from Lemma 7.4, with $\nu = 0$, that the number of zeros of y in the interval $x_1 \leqslant x \leqslant p$ does not exceed

$$\frac{1}{\pi}\int\limits_{x_1}^{p} \{\lambda-q(x)\}^{\frac{1}{2}}\,dx+O(1). \tag{7.5.12}$$

It remains to consider the interval $0 \leqslant x \leqslant x_1$, and here the most natural source of information seems to be (5.4.5). Suppose, for example, that $\eta'(0) = 0$. Then, writing $\eta_1(x) = \eta(x)/\eta(0)$,

$$\eta_1(x) = \cos\xi(x)+\int\limits_0^x \sin\{\xi(x)-\xi(t)\}R(t)\eta_1(t)\,dt.$$

Suppose that

$$\int\limits_0^{x_1} |R(t)|\,dt \leqslant \delta. \tag{7.5.13}$$

Then

$$|\eta_1(x)| \leqslant 1+\int\limits_0^x |R(t)\eta_1(t)|\,dt,$$

$$|\eta_1(x)| \leqslant \exp\left\{\int\limits_0^x |R(t)|\,dt\right\},$$

by Lemma 5.2, and

$$\left|\int\limits_0^x \sin\{\xi(x)-\xi(t)\}R(t)\eta_1(t)\,dt\right| \leqslant \int\limits_0^x |R(t)|e^{\int_0^t |R(t')|\,dt'}\,dt$$

$$= \exp\left\{\int\limits_0^x |R(t)|\,dt\right\}-1$$

$$\leqslant e^{\delta}-1 < \frac{\delta}{1-\delta} < 2\delta,$$

if $\delta < \frac{1}{2}$.

Consider the interval $n\pi \leqslant \xi(x) \leqslant (n+1)\pi$, and suppose, for example, that n is even. If $\delta < 1/(2\surd 2)$, it follows that $\eta_1(x)$ is positive for $n\pi \leqslant \xi \leqslant (n+\frac{1}{4})\pi$ and negative for $(n+\frac{3}{4})\pi \leqslant \xi \leqslant (n+1)\pi$. Hence there is at least one zero in the interval $(n+\frac{1}{4})\pi < \xi < (n+\frac{3}{4})\pi$. Also

$$\eta_1'(x) = -\xi'(x)\sin\xi(x)+\xi'(x)\int\limits_0^x \cos\{\xi(x)-\xi(t)\}R(t)\eta_1(t)\,dt,$$

and a similar argument shows that $\eta_1'(x)$ is negative throughout the interval $(n+\frac{1}{4})\pi < \xi < (n+\frac{3}{4})\pi$. Hence $\eta_1(x)$ decreases steadily, and so has only one zero in the interval. The number of zeros of $\eta(x)$, and so of y, in $(0,x_1)$ is therefore

$$\frac{1}{\pi}\int\limits_0^{x_1}\{\lambda-q(x)\}^{\frac{1}{2}}\,dx+O(1).$$

The theorem then follows from this, (7.5.12), and (7.5.3).

It remains to prove (7.5.13) with $\delta \leqslant 1/(2\surd 2)$ and λ sufficiently large. Now

$$\int\limits_0^{x_1}\frac{q'^2(t)}{\{\lambda-q(t)\}^{\frac{5}{2}}}dt = \frac{2}{3}\frac{q'(x_1)}{\{\lambda-q(x_1)\}^{\frac{3}{2}}}-\frac{2}{3}\frac{q'(0)}{\lambda^{\frac{3}{2}}}+\frac{2}{3}\int\limits_0^{x_1}\frac{q''(t)}{\{\lambda-q(t)\}^{\frac{3}{2}}}\,dt.$$

The first integrated term tends to zero, since

$$\lambda-q(x) \geqslant \{q'(x)\}^{2-\gamma} \quad (x \leqslant x_1).$$

It is therefore sufficient to prove that

$$\int\limits_0^{x_1}\frac{q''(t)}{\{\lambda-q(t)\}^{\frac{3}{2}}}\,dt \to 0.$$

Let $q(x_0) = \frac{1}{2}\lambda$. Then this integral is less than

$$\int\limits_0^{x_0}\frac{q''(t)}{(\frac{1}{2}\lambda)^{\frac{3}{2}}}\,dt+\int\limits_{x_0}^{x_1}\frac{q''(t)}{\{q'(t)\}^{3-3\gamma/2}}\,dt$$

$$= \frac{q'(x_0)-q'(0)}{(\frac{1}{2}\lambda)^{\frac{3}{2}}}+\frac{1}{2-3\gamma/2}\left[\frac{1}{\{q'(x_0)\}^{2-3\gamma/2}}-\frac{1}{\{q'(x_1)\}^{2-3\gamma/2}}\right].$$

This all tends to zero (the first term since $q'(x_0) \leqslant q'(p) = o(\lambda^{\frac{3}{2}})$ by (7.3.5)). The theorem therefore follows.

7.6. The following consequences of Theorem 7.5 will be required later.

THEOREM 7.6. *Under the conditions of Theorem* 7.5,

$$N(\lambda+\surd\lambda)-N(\lambda) = O\{p(\lambda)\}, \tag{7.6.1}$$

$$N(\lambda+\mu)-N(\lambda) = O\{\mu\lambda^{-\frac{1}{2}}p(\lambda)\} \quad (\surd\lambda \leqslant \mu \leqslant \lambda), \tag{7.6.2}$$

$$N(\lambda+\mu) = N(\lambda)\{1+O(\mu/\lambda)\} \quad (\surd\lambda \leqslant \mu \leqslant \lambda). \tag{7.6.3}$$

By Theorem 7.5

$$\pi\{N(\lambda+\surd\lambda)-N(\lambda)\} = \int\limits_0^{p(\frac{1}{2}\lambda)} [\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}-\{\lambda-q(x)\}^{\frac{1}{2}}]\,dx +$$

$$+\int\limits_{p(\frac{1}{2}\lambda)}^{p(\lambda)} [\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}-\{\lambda-q(x)\}^{\frac{1}{2}}]\,dx +$$

$$+\int\limits_{p(\lambda)}^{p(\lambda+\surd\lambda)} \{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}\,dx +O(1)$$

$$= I_1+I_2+I_3+O(1)$$

say. Now

$$\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}-\{\lambda-q(x)\}^{\frac{1}{2}} = \frac{\surd\lambda}{\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}+\{\lambda-q(x)\}^{\frac{1}{2}}}.$$

For $x \leqslant p(\frac{1}{2}\lambda)$, $q(x) \leqslant \frac{1}{2}\lambda$, so that this expression is bounded. Hence

$$I_1 = O\{p(\tfrac{1}{2}\lambda)\} = O\{p(\lambda)\}.$$

Also

$$I_2 = O\left[\int\limits_{p(\frac{1}{2}\lambda)}^{p(\lambda)} \frac{\surd\lambda\,dx}{\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}}\right] = O\left[\frac{\surd\lambda}{q'\{p(\frac{1}{2}\lambda)\}}\int\limits_{p(\frac{1}{2}\lambda)}^{p(\lambda)} \frac{q'(x)\,dx}{\{\lambda+\surd\lambda-q(x)\}^{\frac{1}{2}}}\right]$$

$$= O\left[\frac{\surd\lambda}{q'\{p(\frac{1}{2}\lambda)\}}[\lambda+\surd\lambda-q(x)]_{p(\frac{1}{2}\lambda)}^{p(\lambda)}\right] = O\left[\frac{\lambda}{q'\{p(\frac{1}{2}\lambda)\}}\right].$$

Now

$$q(x) = \int\limits_0^x q'(t)\,dt \leqslant xq'(x).$$

Hence

$$q'\{p(\tfrac{1}{2}\lambda)\} \geqslant \frac{q\{p(\frac{1}{2}\lambda)\}}{p(\frac{1}{2}\lambda)} = \frac{\frac{1}{2}\lambda}{p(\frac{1}{2}\lambda)}.$$

Hence

$$I_2 = O\{p(\tfrac{1}{2}\lambda)\} = O\{p(\lambda)\}.$$

Finally

$$I_3 = O[\lambda^{\frac{1}{4}}\{p(\lambda+\surd\lambda)-p(\lambda)\}] = O\{\lambda^{\frac{3}{4}}p'(\lambda)\}$$

$$= O\left\{\frac{\lambda^{\frac{3}{4}}}{q'\{p(\lambda)\}}\right\} = O\left\{\frac{p(\lambda)}{\lambda^{\frac{1}{4}}}\right\} = O\{p(\lambda)\},$$

and (7.6.1) follows. (7.6.2) follows on applying (7.6.1) $O(\mu\lambda^{-\frac{1}{2}})$ times, and observing that, since $p(x)$ is concave downwards,

$$p(x) \leqslant 2p(\tfrac{1}{2}x).$$

Also (7.6.3) follows from (7.6.2), since by (7.5.3)

$$N(\lambda) > A\lambda^{\frac{1}{2}}p(\tfrac{1}{2}\lambda) > A\lambda^{\frac{1}{2}}p(\lambda).$$

REFERENCES

Milne (1), Langer (1), (2), (3), Titchmarsh (8), (10).

VIII

FURTHER APPROXIMATIONS TO $N(\lambda)$

8.1. The following argument is used by physicists. In quantum mechanics the equation

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}\{E-V\}\psi = 0 \tag{8.1.1}$$

is considered, where $V = V(x)$, and E, m, and h are constants, h being small. Let $\chi(x)$ be defined by

$$\psi = \exp\left\{\frac{2\pi i}{h}\int \chi\, dx\right\}. \tag{8.1.2}$$

Then
$$\frac{d\psi}{dx} = \frac{2\pi i\chi}{h}\psi, \tag{8.1.3}$$

$$\frac{d^2\psi}{dx^2} = \frac{2\pi i\psi}{h}\frac{d\chi}{dx}+\left(\frac{2\pi i\chi}{h}\right)^2\psi.$$

Substituting in (8.1.1), and dividing by ψ, we obtain

$$\frac{2\pi i}{h}\frac{d\chi}{dx}+\left(\frac{2\pi i\chi}{h}\right)^2+\frac{8\pi^2 m}{h^2}(E-V) = 0$$

or
$$\frac{h}{2\pi i}\frac{d\chi}{dx} = 2m(E-V)-\chi^2. \tag{8.1.4}$$

If h were zero, it would follow that

$$\chi = \{2m(E-V)\}^{\frac{1}{2}} = \chi_0 \tag{8.1.5}$$

say. Now suppose that χ is expanded in a series of the form

$$\chi = \chi_0+\frac{h}{2\pi i}\chi_1+\left(\frac{h}{2\pi i}\right)^2\chi_2+\cdots. \tag{8.1.6}$$

Substituting in (8.1.3), and equating to zero coefficients of different powers of y, we obtain equations which determine the χ_n in succession. They are (8.1.5), and

$$\frac{d\chi_{n-1}}{dx} = -\sum_{m=0}^{n}\chi_{n-m}\chi_m \quad (n = 1,\ 2,\ldots). \tag{8.1.7}$$

Thus
$$\chi_1 = -\frac{1}{2}\frac{\chi_0'}{\chi_0} = \frac{1}{4}\frac{V'}{E-V},$$

$$\chi_2 = -\frac{\chi_1^2+\chi_1'}{2\chi_0} = -\frac{1}{2\chi_0}\left\{\frac{V'^2}{16(E-V)^2}+\frac{V'^2}{4(E-V)^2}+\frac{V''}{4(E-V)}\right\}$$
$$= -\frac{5V'^2}{32(2m)^{\frac{1}{2}}(E-V)^{\frac{5}{2}}}-\frac{V''}{8(2m)^{\frac{1}{2}}(E-V)^{\frac{3}{2}}},$$

and so on.

Now apply these formulae to the problem of the relation between n and λ_n. Replace x by a complex variable z, and suppose that $V(z)$ is an analytic function of z, regular in any relevant region. Then any solution $\psi = \psi(z)$ of (8.1.1) is an analytic function of z. Suppose that there is a discrete spectrum, and let $E = E_n$ be the $(n+1)$th eigenvalue. Then $\psi = \psi_n(z)$ has n real zeros. Suppose that there is a contour C which includes these zeros but no others. Then

$$n = \frac{1}{2\pi i}\int\limits_C \frac{\psi_n'(z)}{\psi_n(z)}\,dz = \frac{1}{h}\int\limits_C \chi(z)\,dz$$

by (8.1.3). Hence

$$n = \frac{1}{h}\int\limits_C \chi_0(z)\,dz+\frac{1}{2\pi i}\int\limits_C \chi_1(z)\,dz-\frac{h}{4\pi^2}\int\limits_C \chi_2(z)\,dz+\dots.$$

Let $V(z) = E$ at $z = a$ and $z = b$, and suppose that these are simple zeros of $V(z)-E$. Then $\chi_1(z)$ has simple poles at these points with residue $-\frac{1}{4}$ at each. Hence

$$\frac{1}{2\pi i}\int\limits_C \chi_1(z)\,dz = -\tfrac{1}{2}.$$

Also, integrating by parts,

$$\int\limits_C \frac{V''(z)}{\{E-V(z)\}^{\frac{3}{2}}}\,dz = -\frac{3}{2}\int\limits_C \frac{V'^2(z)}{\{E-V(z)\}^{\frac{5}{2}}}\,dz.$$

Hence
$$\int\limits_C \chi_2(z)\,dz = \frac{1}{32(2m)^{\frac{1}{2}}}\int\limits_C \frac{V'^2(z)}{\{E-V(z)\}^{\frac{5}{2}}}\,dz.$$

Hence we obtain the formula

$$n+\tfrac{1}{2} = \frac{(2m)^{\frac{1}{2}}}{h}\int\limits_C \{E-V(z)\}^{\frac{1}{2}}\,dz-\frac{h}{128\pi^2(2m)^{\frac{1}{2}}}\int\limits_C \frac{V'^2(z)}{\{E-V(z)\}^{\frac{5}{2}}}\,dz+\dots. \tag{8.1.8}$$

All the above analysis is, of course, purely formal. The validity of an expansion of the form (8.1.6) remains to be investigated; and h is not arbitrarily small, but merely a numerical constant. Even the

formal appearance of successive approximations with h small, upon which (8.1.6) depends, vanishes when we come to (8.1.8); for if E is to be an eigenvalue of (8.1.1), it must depend on h. Suppose, for example, that $V(x) = ax^4$. On making the substitutions

$$X = \left(\frac{8\pi^2 ma}{h^2}\right)^{\frac{1}{6}} x, \qquad \lambda = 4E\left(\frac{\pi^4 m^2}{h^4 a}\right)^{\frac{1}{3}},$$

(8.1.1) takes the standard form

$$\frac{d^2\psi}{dX^2}+(\lambda-X^4)\psi = 0. \tag{8.1.9}$$

If λ_0 is the smallest eigenvalue of this, that of (8.1.1) is

$$E_0 = \frac{\lambda_0}{4}\left(\frac{h^4 a}{\pi^4 m^2}\right)^{\frac{1}{3}}.$$

It is easily verified, e.g. by making the substitution $z = h^{\frac{1}{3}}\zeta$, that each term on the right-hand side of (8.1.8) is independent of h.

The above analysis is known as the B.W.K. (or W.B.K. or W.K.B.) method.

8.2. The equation (8.1.1) reduces to the standard form on putting

$$\frac{8\pi^2 mE}{h^2} = \lambda, \qquad \frac{8\pi^2 m}{h^2} V(x) = q(x). \tag{8.2.1}$$

We have therefore to consider the equation

$$\frac{d^2\psi}{dz^2}+\{\lambda-q(z)\}\psi = 0 \tag{8.2.2}$$

in which the usual real variable x is replaced by a complex variable z, and $q(z)$ is an analytic function of z, regular in a certain domain containing $z = 0$. The argument of § 1.5 is easily adapted to show that, if $\psi(z)$ satisfies given boundary conditions at $z = 0$, it is an analytic function of z, for each λ, regular in the same domain as $q(z)$. Since the solutions to be considered are eigenfunctions, λ may be supposed real.

The situation seems to depend a good deal on the nature of the function $q(z)$. In order to obtain precise results, we consider the special case $q(z) = z^k$, where k is an even positive integer; but it will be seen that it is possible to extend the method to any analytic $q(z)$ which mimics this special case sufficiently closely.

8.3. It is a question of obtaining approximations to solutions of (8.2.2) in certain regions of the z-plane.

Let λ be real and positive, and let p denote the real positive value of $\lambda^{1/k}$. Let

$$\xi(z) = \int\limits_0^z (\lambda - w^k)^{\frac{1}{2}}\,dw, \qquad \eta(z) = (\lambda - z^k)^{\frac{1}{4}}\psi_n(z), \tag{8.3.1}$$

where $(\lambda-w^k)^{\frac{1}{2}}$ and $(\lambda-z^k)^{\frac{1}{4}}$ reduce to the real positive values of $\lambda^{\frac{1}{2}}$ and $\lambda^{\frac{1}{4}}$ as $w \to 0$, $z \to 0$. As w passes above p from a real value less than p to a real value greater than p, $\arg(\lambda - w^k)$ decreases from 0 to $-\pi$. Hence $(\lambda-w^k)^{\frac{1}{2}}$ becomes $-i(w^k-\lambda)^{\frac{1}{2}}$, where the square root is real and positive. Hence $e^{i\xi(z)}$ becomes exponentially large as $z \to \infty$ along the real axis.

The analysis of § 5.4 obviously extends to complex variables, so that $\eta(z)$ satisfies the integral equation

$$\eta(z) = \eta(0)\cos\xi(z) + \eta'(0)\lambda^{-\frac{1}{2}}\sin\xi(z) +$$
$$+ \int\limits_0^z \sin\{\xi(z)-\xi(w)\}R(w)\eta(w)\,dw, \tag{8.3.2}$$

where

$$\begin{aligned} R(w) &= -\frac{1}{4}\frac{q''(w)}{\{\lambda-q(w)\}^{\frac{3}{2}}} - \frac{5}{16}\frac{q'^2(w)}{\{\lambda-q(w)\}^{\frac{5}{2}}} \\ &= -\frac{k(k-1)w^{k-2}}{4(\lambda-w^k)^{\frac{3}{2}}} - \frac{5}{16}\frac{k^2w^{2k-2}}{(\lambda-w^k)^{\frac{5}{2}}} \\ &= -\frac{kw^{k-2}\{4(k-1)\lambda+(k+4)w^k\}}{16(\lambda-w^k)^{\frac{5}{2}}}. \end{aligned}$$

Consider first the region $\pi/k \leqslant \arg z \leqslant \frac{1}{2}\pi$, $|z| \leqslant p(\sin\pi/k)^{1/k}$. Then $|z|^k \leqslant \lambda\sin\pi/k$, and

$$-\frac{\pi}{k} \leqslant \arg(\lambda - z^k) \leqslant \frac{\pi}{k}$$

for all z in this region. Taking the integral defining $\xi(z)$ along a straight line, so that $\pi/k \leqslant \arg dw \leqslant \frac{1}{2}\pi$, it follows that

$$\frac{\pi}{2k} \leqslant \arg\{(\lambda-w^k)^{\frac{1}{2}}\,dw\} \leqslant \frac{\pi}{2}+\frac{\pi}{2k}.$$

Hence
$$\mathbf{I}\{\xi(z)\} \geqslant \sin\frac{\pi}{2k}\int\limits_0^z |(\lambda-w^k)^{\frac{1}{2}}\,dw| > K\lambda^{\frac{1}{2}}|z|,$$

where K denotes a positive number depending on k only.

Putting $\eta(z) = e^{-i\xi(z)}H(z)$ in (8.3.2),

$$H(z) = \eta(0)e^{i\xi(z)}\cos\xi(z)+\lambda^{-\frac{1}{2}}\eta'(0)e^{i\xi(z)}\sin\xi(z)+$$

$$+\int\limits_0^z e^{i\{\xi(z)-\xi(w)\}}\sin\{\xi(z)-\xi(w)\}R(w)H(w)\,dw.$$

If n is even, $\psi_n(z)$ is an even function, $\eta'(0) = 0$, and this gives

$$|H(z)| \leqslant |\eta(0)|+\int\limits_0^z |R(w)H(w)\,dw|.$$

Hence by Lemma 5.2

$$|H(z)| \leqslant |\eta(0)|\exp\left\{\int\limits_0^z |R(w)\,dw|\right\}.$$

It then follows from (8.3.2) that

$$\eta(z) = \eta(0)\cos\xi(z)+e^{-i\xi(z)}\chi(z),$$

where

$$|\chi(z)| \leqslant |\eta(0)|\left[\int\limits_0^z |R(w)|\exp\left\{\int\limits_0^w |R(w')\,dw'|\right\}|dw|\right]$$

$$= |\eta(0)|\left[\exp\left\{\int\limits_0^z |R(w)\,dw|\right\}-1\right].$$

Hence
$$\eta(z) = \tfrac{1}{2}\eta(0)e^{-i\xi(z)}\{1+\omega(z)\},$$

where

$$|\omega(z)| \leqslant |e^{2i\xi(z)}|+2|\chi(z)|/|\eta(0)|$$

$$\leqslant |e^{2i\xi(z)}|+2\left\{\exp\int\limits_0^z |R(w)\,dw|-1\right\}. \tag{8.3.3}$$

Now
$$\int\limits_0^z |R(w)\,dw| = O\left(\int\limits_0^p \frac{\lambda\rho^{k-2}+\rho^{2k-2}}{\lambda^{\frac{3}{2}}}\,d\rho\right) = O(\lambda^{-\frac{1}{2}-1/k}).$$

If we also assume that $|z| > Kp$, it follows that

$$\omega(z) = O(\lambda^{-\frac{1}{2}-1/k}). \tag{8.3.4}$$

The result is that, in the region considered,

$$\psi_n(z) = \tfrac{1}{2}\psi_n(0)\frac{\lambda_n^{\frac{1}{4}}}{(\lambda_n-z^k)^{\frac{1}{4}}}e^{-i\xi(z)}\{1+\omega(z)\}. \tag{8.3.5}$$

8.4. The above result holds in particular on the segment of the straight line $\mathbf{I}(z) = \frac{1}{2}p\tan\pi/k$ between the imaginary axis and $\arg z = \pi/k$. We next require a similar result on the same line as

far as $\mathbf{R}(z) = p$, and then round the circle $|z-p| = \frac{1}{2}p\tan\pi/k$ as far as the real axis. Let C denote this curve, together with its reflections in the real and imaginary axes.

Since

$$\int_0^z e^{i\{\xi(w)-\xi(z)\}}R(w)\eta(w)\,dw = e^{-i\xi(z)}\int_0^{z_1} e^{i\xi(w)}R(w)\eta(w)\,dw + \\ + \int_{z_1}^z e^{i\{\xi(w)-\xi(z)\}}R(w)\eta(w)\,dw$$

and

$$\int_0^z e^{i\{\xi(z)-\xi(w)\}}R(w)\eta(w)\,dw = e^{i\xi(z)}\int_0^\infty e^{-i\xi(w)}R(w)\eta(w)\,dw - \\ - \int_z^\infty e^{i\{\xi(z)-\xi(w)\}}R(w)\eta(w)\,dw,$$

(8.3.2) is formally (i.e. apart from the question of the convergence of the integrals at infinity) equivalent to

$$\eta(z) = Ae^{i\xi(z)}+Be^{-i\xi(z)}+\tfrac{1}{2}i\int_{z_1}^z e^{i\{\xi(w)-\xi(z)\}}R(w)\eta(w)\,dw + \\ +\tfrac{1}{2}i\int_z^\infty e^{i\{\xi(z)-\xi(w)\}}R(w)\eta(w)\,dw, \quad (8.4.1)$$

where A, B, and z_1 are independent of z.

Let Γ be the region defined by

$$0 \leqslant \mathbf{I}(z) \leqslant \tfrac{1}{2}p\tan\pi/k, \qquad |z-p| \geqslant \tfrac{1}{2}p\tan\pi/k, \qquad \mathbf{R}(z) > 0,$$

together with the segment of straight line $\mathbf{I}(z) = \frac{1}{2}p$, $\frac{1}{2}p \leqslant \mathbf{R}(z) \leqslant p$. Let z_1 be the left-hand end-point of this segment, z any point of Γ. Take $A = 0$, $B = 1$ in (8.4.1), and let the path from z_1 to z consist of the curve C as far as the straight line through z parallel to the real axis, and then this straight line. Let the path from z to infinity be a straight line parallel to the real axis.

If w is in Γ, $|\lambda-w^k| > K|w|^k$, where K depends on k only. Hence

$$|R(w)| < \frac{Kp^k}{|w|^{\frac{3}{2}k+2}}+\frac{K}{|w|^{\frac{1}{2}k+2}} < \frac{K}{|w|^{\frac{1}{2}k+2}}$$

since $|w| > Kp$. Hence

$$\int_z^\infty |R(w)\,dw| = O\left(\int_{Kp}^\infty \frac{du}{u^{\frac{1}{2}k+2}}\right) = O\left(\frac{1}{p^{\frac{1}{2}k+1}}\right).$$

A similar argument applies to the rectilinear part of the path (z_1, z), and the circular part is of length $O(p)$, so that the integral of $|R(w)|$ round this is also $O(p^{-\frac{1}{2}k-1})$. Hence, if δ denotes the maximum of

$$\tfrac{1}{2}\int\limits_{z_1}^{z} |R(w)\,dw| + \tfrac{1}{2}\int\limits_{z}^{\infty} |R(w)\,dw|$$

formed in the above manner, we have

$$\delta = O(\lambda^{-\frac{1}{2}-1/k}).$$

Thus $\delta < 1$ if λ is large enough.

Now take $A = 0$, $B = 1$ in (8.4.1), and form a solution by iteration. Let

$$\eta_1(z) = e^{-i\xi(z)}$$

and

$$\eta_{\nu+1}(z) = e^{-i\xi(z)} + \tfrac{1}{2}i\int\limits_{z_1}^{z} e^{i\{\xi(w)-\xi(z)\}}R(w)\eta_\nu(w)\,dw + $$
$$+\tfrac{1}{2}i\int\limits_{z}^{\infty} e^{i\{\xi(z)-\xi(w)\}}R(w)\eta_\nu(w)\,dw$$

for $\nu \geqslant 1$. Then

$$\eta_2(z)-\eta_1(z) = \tfrac{1}{2}ie^{-i\xi(z)}\left\{\int\limits_{z_1}^{z} R(w)\,dw + \int\limits_{z}^{\infty} e^{2i\{\xi(z)-\xi(w)\}}R(w)\,dw\right\}.$$

Now $$i\{\xi(z)-\xi(w)\} = -i\int\limits_{z}^{w} (\lambda-w'^k)^{\frac{1}{2}}\,dw',$$

where $\lambda-w'^k$ lies in the lower half-plane, and so $(\lambda-w'^k)^{\frac{1}{2}}$ lies in the fourth quadrant (since it reduces to $\lambda^{\frac{1}{2}}$ as $w' \to 0$); and dw' is real and positive. Hence

$$\mathbf{R}[i\{\xi(z)-\xi(w)\}] \leqslant 0,$$
$$|e^{2i\{\xi(z)-\xi(w)\}}| \leqslant 1.$$

Hence $$|\eta_2(z)-\eta_1(z)| \leqslant \delta e^{\mathbf{I}\{\xi(z)\}}.$$

Similarly

$$\eta_3(z)-\eta_2(z) = \tfrac{1}{2}i\int\limits_{z_1}^{z} e^{i\{\xi(w)-\xi(z)\}}R(w)\{\eta_2(w)-\eta_1(w)\}\,dw + $$
$$+\tfrac{1}{2}i\int\limits_{z}^{\infty} e^{i\{\xi(z)-\xi(w)\}}R(w)\{\eta_2(w)-\eta_1(w)\}\,dw,$$

$$|\eta_3(z)-\eta_2(z)| \leqslant \delta^2 e^{\mathbf{I}\{\xi(z)\}},$$

and so on generally. Hence $\sum \{\eta_{\nu+1}(z)-\eta_\nu(z)\}$ is convergent, i.e. $\eta_\nu(z)$

tends to a limit $\eta(z)$, which (as in the case of the similar formulae of § 6.2) satisfies (8.4.1) with $A = 0$, $B = 1$. Hence the corresponding $\psi(z)$ satisfies (8.2.2). If λ is an eigenvalue λ_n, this $\psi(z)$ must be a multiple of the eigenfunction $\psi_n(z)$, since any other solution of the equation tends to infinity as $z \to \infty$ along the real axis.

Now

$$\eta(z) = \eta_1(z)+\sum_{\nu=1}^{\infty}\{\eta_{\nu+1}(z)-\eta_\nu(z)\} = e^{-i\xi(z)}\{1+\chi_1(z)\}$$

say, where

$$|\chi_1(z)| \leqslant \sum_{\nu=1}^{\infty}\delta^n = \frac{\delta}{1-\delta} = O(\lambda^{-\frac{1}{2}-1/2k}). \tag{8.4.2}$$

The result is therefore

$$\psi_n(z) = \frac{c_n\, e^{-i\xi(z)}}{(\lambda_n-z^k)^{\frac{1}{4}}}\{1+\chi_1(z)\}, \tag{8.4.3}$$

where c_n is a constant.

8.5. Let z_0 and z_2 denote the points of C on the positive real and imaginary axes respectively, and z_1 the point defined above.

The function $\psi_n(z)$ is either odd or even—suppose, for example, that it is even. Let N be the number of zeros of $\psi_n(z)$ inside C. Then $2\pi iN$ is equal to the variation of $\log\psi_n(z)$ round C, and so to twice the variation of $\log\psi_n(z)$ round the upper half of C. Hence

$$N = \frac{1}{\pi}\mathbf{I}\{\log\psi_n(-z_0)-\log\psi_n(z_0)\}.$$

The values taken by $\psi_n(z)$ between z_2 and $-z_0$ are the conjugates of those taken between z_2 and z_0. Hence $\log\psi_n(-z_0)-\log\psi_n(z_2)$ is the conjugate of $\log\psi_n(z_0)-\log\psi_n(z_2)$, and

$$\mathbf{I}\{\log\psi_n(-z_0)-\log\psi_n(z_2)\} = \mathbf{I}\{\log\psi_n(z_2)-\log\psi_n(z_0)\}.$$

Hence

$$N = \frac{2}{\pi}\mathbf{I}\{\log\psi_n(z_2)-\log\psi_n(z_0)\}. \tag{8.5.1}$$

Hence by (8.3.5) and (8.4.3)

$$N = \frac{2}{\pi}\mathbf{I}\left\{-i\xi(z_2)+i\xi(z_0)-\tfrac{1}{4}\log\frac{\lambda-z_2^k}{\lambda-z_0^k}+\right.$$
$$\left.+\log\frac{1+\omega(z_2)}{1+\omega(z_1)}+\log\frac{1+\chi_1(z_1)}{1+\chi_1(z_0)}\right\}. \tag{8.5.2}$$

Now

$$-i\xi(z_2)+i\xi(z_1) = -\tfrac{1}{4}i\int\limits_C (\lambda-w^k)^{\frac{1}{2}}\,dw$$
$$= \tfrac{1}{2}i\int\limits_{-p}^{p} (\lambda-w^k)^{\frac{1}{2}}\,dw = i\int\limits_{0}^{p} (\lambda-w^k)^{\frac{1}{2}}\,dw.$$

Next, as z varies from z_0 to z_2, z^k describes a spiral, starting to the right of λ and ending to the left of it, and not encircling it again. Hence $\arg(z^k-\lambda)$ increases by π. Lastly, if λ is so large that $|\omega(z)| < 1$, $|\chi_1(z)| < 1$ on C, $\log\{1+\omega(z)\}$ and $\log\{1+\chi_1(z)\}$ may be assigned their principal values; hence

$$\log\{1+\omega(z)\} = O(\lambda^{-\frac{1}{2}-1/k}), \qquad \log\{1+\chi_1(z)\} = O(\lambda^{-\frac{1}{2}-1/k}).$$

Hence (8.5.2) gives

$$N = \frac{2}{\pi}\int\limits_0^p (\lambda-w^k)^{\frac{1}{2}}\,dw - \tfrac{1}{2}+O(\lambda^{-\frac{1}{2}-1/k}). \tag{8.5.3}$$

Now $\psi_n(z)$ has n real zeros, and by (7.2.3)

$$n \geqslant \frac{2}{\pi}\int\limits_0^p (\lambda-w^k)^{\frac{1}{2}}\,dw - 1.$$

Also $N = n+2m$, where m is zero or a positive integer, since complex zeros occur in conjugate pairs. Hence

$$2m \leqslant \tfrac{1}{2}+O(\lambda^{-\frac{1}{2}-1/k}).$$

If λ is so large that the last term is less than $\frac{3}{2}$, then $m = 0$, $N = n$, so that there are no complex zeros inside the curve C. Replacing N by n in (8.5.3), we obtain a formula equivalent to (8.1.8), with the remainder after the first term on the right-hand side replaced by an O-term.

8.6. Further approximations. The next approximation to the function $\eta(z)$ $(0 \leqslant \arg z \leqslant \pi/k)$ is

$$\eta(z) = \eta_2(z)\{1+\chi_2(z)\},$$

where

$$\eta_2(z) = e^{-i\xi(z)}+\tfrac{1}{2}ie^{-i\xi(z)}\int\limits_{z_1}^{z} R(w)\,dw + \tfrac{1}{2}ie^{i\xi(z)}\int\limits_{z}^{\infty} e^{-2i\xi(w)}R(w)\,dw,$$

and

$$\chi_2(z) = O(\delta^2) = O(\lambda^{-1-2/k}).$$

Now

$$\int\limits_z^\infty e^{-2i\xi(w)}R(w)\,dw = \frac{e^{-2i\xi(z)}R(z)}{2i(\lambda-z^k)^{\frac{1}{2}}}+\int\limits_z^\infty \frac{e^{-2i\xi(w)}}{2i}\frac{d}{dw}\left\{\frac{R(w)}{(\lambda-w^k)^{\frac{1}{2}}}\right\}dw$$

$$= O\{\lambda^{-1-2/k}|e^{-2i\xi(z)}|\}+O\left\{|e^{-2i\xi(z)}|\int\limits_z^\infty\left|\frac{dw}{w^{k+3}}\right|\right\}$$

$$= O\{\lambda^{-1-2/k}|e^{-2i\xi(z)}|\}.$$

Hence
$$\eta(z) = e^{-i\xi(z)}\left\{1+\tfrac{1}{2}i\int\limits_{z_1}^{z} R(w)\,dw + O(\lambda^{-1-2/k})\right\}.$$

A similar formula may be obtained in the angle $\pi/k \leqslant \arg z \leqslant \frac{1}{2}\pi$. It follows that the term $O(\lambda^{-\frac{1}{2}-1/k})$ in (8.5.1) may be replaced by
$$\frac{1}{4\pi}\int\limits_{C} R(w)\,dw + O(\lambda^{-1-2/k}).$$

This corresponds to (8.1.8), with the remainder after the second term on the right-hand side replaced by an O-term. We can plainly proceed indefinitely in the same way. The formula (8.1.8) is therefore justified for the functions $q(z) = z^k$, or for any analytic functions with sufficiently similar properties, provided that the eigenvalue is large enough.

8.7. The case $k = 4$. Here
$$R(w) = -\frac{w^2(3p^4+2w^4)}{(p^4-w^4)^{\frac{5}{2}}}.$$

Consider first the formulae of § 8.4. The curve C now consists of the right-hand semicircle of $|z-p| = \frac{1}{2}p$, and the left-hand semicircle of $|z+p| = \frac{1}{2}p$, joined by segments of the lines $\mathbf{I}(z) = \pm\frac{1}{2}p$. On the upper half of the right-hand semicircle,
$$|p-w| = \tfrac{1}{2}p, \qquad |p+w| > 2p, \qquad |p^2+w^2| > 2p^2$$
and $|w| < \frac{3}{2}p$. Hence
$$|R(w)| < \frac{3p^4(9p^2/4)+2(3p/2)^6}{2^{\frac{5}{2}}p^{10}} = \frac{27\sqrt{2}}{32}\left(1+\frac{27}{8}\right)\frac{1}{p^4} \sim \frac{5{\cdot}2}{p^4}.$$

Integrating over a length $\frac{1}{4}\pi p$, we obtain approximately $4/p^3$.

On the straight line $w = u+\frac{1}{2}ip$, $\frac{1}{2}p \leqslant u \leqslant p$, we have
$$|w|^2 \leqslant 5p^2/4,$$
$$|p+w|^2 = (u+p)^2+\tfrac{1}{4}p^2 \geqslant \left(\frac{3p}{2}\right)^2+\tfrac{1}{4}p^2 = \frac{5p^2}{2}$$
and
$$\begin{aligned}|p^2+w^2|^2 &= |p^2+u^2+ipu-\tfrac{1}{4}p^2|^2\\ &= (u^2+\tfrac{3}{4}p^2)^2+p^2u^2\\ &\geqslant p^4+\tfrac{1}{4}p^4 = \frac{5p^4}{4}.\end{aligned}$$

Hence

$$|R(w)| \leqslant \frac{3p^4\dfrac{5p^2}{4}+2\left(\dfrac{5p^2}{4}\right)^3}{\left(\dfrac{25p^6}{8}\right)^{\frac{5}{4}}(\frac{1}{2}p)^{\frac{5}{2}}}$$

$$= \left(\frac{3.2^{\frac{17}{4}}}{5^{\frac{3}{2}}}+5^{\frac{1}{2}}.2^{\frac{5}{4}}\right)\frac{1}{p^4}$$

$$= 5^{\frac{1}{2}}.2^{\frac{1}{4}}\left(\frac{3.16}{25}+2\right)\frac{1}{p^4}$$

$$\leqslant 5^{\frac{1}{2}}.2^{\frac{1}{4}}.3\,\frac{1}{p^4} = \frac{8}{p^4}$$

approximately. Integrating, we get again $4/p^3$; hence

$$\int\limits_{z_0}^{z_1} |R(w)|\, dw < \frac{8}{p^3}.$$

On the straight line from z to infinity, where z is on or to the right of the semicircle, we have

$$|p+w| \geqslant |w|, \qquad |p^2+w^2| \geqslant |w|^2, \qquad |w-p| \geqslant \tfrac{1}{2}p.$$

Hence $$|R(w)| \leqslant \frac{3p^4|w|^2+2|w|^6}{|w|^{\frac{15}{2}}(\frac{1}{2}p)^{\frac{5}{2}}} = \frac{3.2^{\frac{5}{2}}p^{\frac{3}{2}}}{|w|^{\frac{11}{2}}}+\frac{2^{\frac{7}{2}}}{p^{\frac{5}{2}}|w|^{\frac{3}{2}}}.$$

Hence this part is

$$\leqslant \int\limits_p^\infty \left(\frac{3.2^{\frac{5}{2}}p^{\frac{3}{2}}}{u^{\frac{11}{2}}}+\frac{2^{\frac{7}{2}}}{p^{\frac{5}{2}}u^{\frac{3}{2}}}\right) du$$

$$= (3.2^{\frac{5}{2}}.\tfrac{2}{9}+2^{\frac{7}{2}}.2)\frac{1}{p^3} = (\tfrac{8}{3}+16)\frac{\sqrt{2}}{p^3} = \frac{26}{p^3}$$

approximately. Hence $$\delta < \frac{17}{p^3}.$$

For $\delta < 1$ we therefore want

$$\lambda > 17^{\frac{4}{3}} = 44$$

roughly.

In § 8.3, we have $|w| \leqslant 2^{-\frac{1}{2}}p$, so that

$$|R(w)| \leqslant \frac{3p^4\rho^2+2\rho^6}{(3p^4/4)^{\frac{5}{2}}}.$$

Hence $$\int\limits_0^z |R(w)\, dw| \leqslant \left(\frac{4}{3p^4}\right)^{\frac{5}{2}} \int\limits_0^{p/\sqrt{2}} (3p^4\rho^2+2\rho^6)\, d\rho,$$

which is found to be less than $1/p^3$, and so it is small if λ satisfies the above condition. A closer examination of the above integrals would no doubt show that the method was valid for much smaller values of λ.

8.8. General character of the functions $\psi_n(z)$. If $q(z)$ is an integral function of z, so is $\psi_n(z)$ for each n. From (8.3.2) it follows that

$$\eta(z) = O[\exp|\mathbf{I}\{\xi(z)\}|],$$

and hence that

$$\psi_n(z) = O[|q(z)|^{-\frac{1}{4}}\exp|\mathbf{I}\{\xi(z)\}|].$$

For $q(z) = z^k$,

$$\xi(z) = -i\int\limits_0^z (w^k-\lambda_n)^{\frac{1}{2}}\,dw$$

$$= -i\int\limits_{z_0}^z \left(w^{\frac{1}{2}k}-\frac{\lambda_n}{2w^{\frac{1}{2}k}}+...\right)dw+O(1)$$

$$= -\tfrac{1}{2}iz^2+\tfrac{1}{2}\lambda_n\log z+O(1) \quad (k=2)$$

$$= -i\frac{z^{\frac{1}{2}k+1}}{\frac{1}{2}k+1}+O(1) \quad (k>2).$$

Hence

$$\psi_n(z) = O\{|z|^{\frac{1}{2}\lambda_n-\frac{1}{2}}\exp(\tfrac{1}{2}|z|^2)\} \quad (k=2),$$

$$\psi_n(z) = O\left\{|z|^{-\frac{1}{4}k}\exp\left(\frac{|z|^{\frac{1}{2}k+1}}{\frac{1}{2}k+1}\right)\right\} \quad (k>2).$$

Hence $\psi_n(z)$ is an integral function of order $\frac{1}{2}k+1$ at most. Since by (8.4.3)

$$\psi_n(x) = O\left\{x^K\exp\left(-\frac{x^{\frac{1}{2}k+1}}{\frac{1}{2}k+1}\right)\right\}$$

for real x, the order is exactly $\frac{1}{2}k+1$. Hence $\psi_n(\surd z)$ or $\surd z\,\psi_n(\surd z)$ is an integral function of order $\frac{1}{4}k+\frac{1}{2}$, according to whether n is even or odd.

In the case $k=2$, (8.4.3) holds for all sufficiently large values of z in the first quadrant. The curve C can be replaced by any circle $|z| = R$ with $R > \lambda_n^{\frac{1}{2}}$. Consequently $\psi_n(z)$ has no complex zeros, if n is sufficiently large. Actually this is well known to be true for all values of n.

In the case $k=4$, $\psi_n(\surd z)$ or $\surd z\,\psi_n(\surd z)$ is an integral function of order $\frac{3}{2}$, and so has an infinity of zeros, all but a finite number

of which must be complex. An infinity of these zeros are purely imaginary; for if $z = iy$, (8.2.2) becomes

$$\frac{d^2\psi}{dy^2}+(y^4-\lambda)\psi = 0. \tag{8.8.1}$$

The coefficient of ψ is positive for $y > \lambda^{\frac{1}{4}}$, indicating that the solution is oscillatory. For $y \geqslant (2\lambda)^{\frac{1}{4}}$, $y^4-\lambda \geqslant \frac{1}{2}y^4$. Hence (8.8.1) has a zero between any two consecutive zeros of a solution of

$$\frac{d^2\psi}{dy^2}+\tfrac{1}{2}y^4\psi = 0,$$

e.g. of

$$y^{\frac{1}{2}}J_{\frac{1}{6}}\left(\frac{y^3}{3\sqrt{2}}\right).$$

It is possible that in this case all the zeros of $\psi_n(z)$ are either purely real or purely imaginary. For $k > 4$ the situation is probably still more complicated.

REFERENCES

Brillouin (1), Wentzel (1), Kramers (1), Bell (1), Birkhoff (3), Dunham (1), (2), Kemble (2), Langer (4), (6).

The method employed here is perhaps new.

IX

CONVERGENCE OF THE SERIES EXPANSION UNDER FOURIER CONDITIONS

9.1. The problem of this chapter is similar to that of Chapter VI, but it is now assumed that $q(x) \to \infty$. The expansion thus takes the series form

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x). \tag{9.1.1}$$

It will now be shown that, by imposing further conditions on $q(x)$, we can relax those which were imposed on $f(x)$ in Chapter II; and in fact that, for a wide class of functions $q(x)$, the expansion converges if $f(x)$ satisfies conditions similar to those which are sufficient for the convergence of an ordinary Fourier series.

As in Chapter VI, it is mainly a question of obtaining asymptotic formulae for the functions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ for large complex values of λ; these then give what is required for the function

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy + \phi(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy. \tag{9.1.2}$$

We obtain satisfactory results provided that λ is not too near to the positive real axis. A restriction of this kind is to be expected, since it is here that $\Phi(x,\lambda)$ must have an infinity of poles. However, in the neighbourhood of the poles, the formula

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n \psi_n(x)}{\lambda-\lambda_n} \tag{9.1.3}$$

can be used. For this purpose we have to know something about the distribution of the eigenvalues λ_n, and the analysis of Chapter VII is needed.

9.2. It will be assumed that $q(x)$ is twice differentiable,

$$q'(x) > 0, \quad q''(x) \geqslant 0, \quad \text{and} \quad q''(x) \leqslant \{q'(x)\}^\gamma$$

for sufficiently large x, where $1 < \gamma < \frac{4}{3}$. Then Theorems 7.5 and 7.6 hold.

The analysis of § 5.4 is required again here. With the notation there used and the conditions assumed above, we have

LEMMA 9.2. $$\int_0^\infty |R(t)|\,dt \to 0$$

as $\lambda \to \infty$, it being supposed that $\lambda = u+iv$, $v \geqslant 0$, and λ does not enter the region between the positive real axis and the curve $v = u^{2/\{3(2-\gamma)\}}\chi(u)$; $\chi(u)$ being any continuous function which tends steadily to infinity with u.

The conditions assumed imply that
$$q'(x) = O[\{q(x)\}^c],$$
where $c = 1/(2-\gamma)$, by (7.3.4). Hence
$$\int_0^\infty \frac{q'^2(t)\,dt}{|\lambda-q(t)|^{\frac{5}{2}}} = O\left\{\int_0^\infty \frac{\{q(t)\}^c q'(t)}{|\lambda-q(t)|^{\frac{5}{2}}}\,dt\right\} = O\left\{\int_0^\infty \frac{q^c\,dq}{\{(u-q)^2+v^2\}^{\frac{5}{4}}}\right\}.$$
If $u \geqslant 0$, put $q = u+vr$. We obtain
$$O\left\{\int_{-u/v}^\infty \frac{(u+vr)^c v\,dr}{(v^2r^2+v^2)^{\frac{5}{4}}}\right\} = O\left(\frac{u^c+v^c}{v^{\frac{3}{2}}}\right).$$
If $u < 0$, the integral is
$$O\left\{\int_0^\infty \frac{q^c\,dq}{(|\lambda|^2+q^2)^{\frac{5}{4}}}\right\} = O(|\lambda|^{c-\frac{3}{2}}).$$
Since $q''(t) = O[\{q'(t)\}^\gamma] = O[\{q(t)\}^{c-1}q'(t)]$,
$$\int_0^\infty \frac{q''(t)\,dt}{|\lambda-q(t)|^{\frac{3}{2}}} = O\left\{\int_0^\infty \frac{q^{c-1}\,dq}{|\lambda-q|^{\frac{3}{2}}}\right\}$$
and similar results hold for this integral. The lemma clearly follows from these results.

9.3. Let $\phi(x,\lambda)$ denote again the solution of
$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0 \tag{9.3.1}$$
such that $\phi(0,\lambda) = \sin\alpha$, $\phi'(0,\lambda) = -\cos\alpha$. For a fixed x, or x in a fixed finite range, Lemma 2 of § 1.7 gives
$$\phi(x,\lambda) = \cos(x\surd\lambda)\{\sin\alpha+O(|\lambda|^{-\frac{1}{2}})\} \tag{9.3.2}$$
if $\sin\alpha \neq 0$, and
$$\phi(x,\lambda) = -\frac{\sin(x\surd\lambda)}{\surd\lambda}\{\cos\alpha+O(|\lambda|^{-\frac{1}{2}})\} \tag{9.3.3}$$
if $\sin\alpha = 0$.

For $x \to \infty$, the relevant formulae are (5.6.1), (5.6.2). Also by (5.6.1) and Lemma 9.2, the integral on the right of (5.3.6) is

$$O\left\{|\lambda|^{\frac{1}{4}} \int\limits_0^\infty |R(t)|\, dt\right\} = o(|\lambda|^{\frac{1}{4}})$$

as $\lambda \to \infty$ in the region allowed by the lemma. Hence

$$\phi(x,\lambda) \sim \frac{M(\lambda)e^{-i\xi(x)}}{\{\lambda - q(x)\}^{\frac{1}{4}}}, \tag{9.3.4}$$

and

$$M(\lambda) \sim \tfrac{1}{2}\lambda^{\frac{1}{4}} \sin\alpha \tag{9.3.5}$$

as $\lambda \to \infty$ in the above region, provided that $\sin\alpha \neq 0$. Similarly, if $\sin\alpha = 0$,

$$M(\lambda) \sim -\tfrac{1}{2}i\,\frac{\cos\alpha}{\lambda^{\frac{1}{4}}}. \tag{9.3.6}$$

The formula obtained by differentiating (5.6.2),

$$\frac{d}{dx}[\{\lambda - q(x)\}^{\frac{1}{4}}\phi(x,\lambda)] \sim -iM(\lambda)\{\lambda - q(x)\}^{\frac{1}{4}}e^{-i\xi(x)}, \tag{9.3.7}$$

is also easily proved by first differentiating (5.4.5).

9.4. It is now a question of obtaining a solution of (9.3.1) which is small for large x. We proceed as in § 6.2 or § 8.3, but $e^{i\xi(x)}$ is now small for large x, not large as in § 8.3. Hence (5.4.5) is formally equivalent to

$$\eta(x) = \tfrac{1}{2}\eta(0)\{e^{i\xi(x)}+e^{-i\xi(x)}\} - \tfrac{1}{2}i\eta'(0)\lambda^{-\frac{1}{2}}\{e^{i\xi(x)} - e^{-i\xi(x)}\} +$$
$$+\frac{1}{2i}\int\limits_0^x e^{i\{\xi(x)-\xi(t)\}}R(t)\eta(t)\,dt + \frac{1}{2i}\int\limits_x^\infty e^{i\{\xi(t)-\xi(x)\}}R(t)\eta(t)\,dt -$$
$$-\frac{1}{2i}\int\limits_0^\infty e^{i\{\xi(t)-\xi(x)\}}R(t)\eta(t)\,dt$$

or

$$\eta(x) = Ae^{i\xi(x)} + Be^{-i\xi(x)} + \frac{1}{2i}\int\limits_0^x e^{i\{\xi(x)-\xi(t)\}}R(t)\eta(t)\,dt +$$
$$+\frac{1}{2i}\int\limits_x^\infty e^{i\{\xi(t)-\xi(x)\}}R(t)\eta(t)\,dt. \tag{9.4.1}$$

Take $A = 1$, $B = 0$, $\eta_1 = e^{i\xi(x)}$, and for $n \geqslant 1$

$$\eta_{n+1}(x) = e^{i\xi(x)} + \frac{1}{2i}\int_0^x e^{i\{\xi(x)-\xi(t)\}}R(t)\eta_n(t)\,dt + $$
$$+\frac{1}{2i}\int_x^\infty e^{i\{\xi(t)-\xi(x)\}}R(t)\eta_n(t)\,dt.$$

Let J now denote $\int_0^\infty |R(t)|\,dt$; by the above lemma, $J < 2$ if λ is sufficiently large and in the above region. Arguing as in § 6.2, it follows that $\eta_n(x)$ tends to a limit $\eta(x)$, which satisfies (9.4.1) with $A = 0$, $B = 1$; and

$$|\eta(x)| \leqslant \frac{|e^{i\xi(x)}|}{1-\frac{1}{2}J}, \tag{9.4.2}$$

$$|\eta(x)-e^{i\xi(x)}| \leqslant \frac{\frac{1}{2}J}{1-\frac{1}{2}J}|e^{i\xi(x)}|. \tag{9.4.3}$$

In particular, $y(x) = \eta(x)\{\lambda-q(x)\}^{-\frac{1}{4}}$ is $L^2(0,\infty)$.

9.5. We have to apply this result to the function $\psi(x,\lambda)$. Now

$$\psi(x,\lambda) = A(\lambda)\phi(x,\lambda)+B(\lambda)y(x,\lambda)$$

and $\psi(x,\lambda)$ and $y(x,\lambda)$ are $L^2(0,\infty)$, while $\phi(x,\lambda)$ is not, at any rate if $M(\lambda) \neq 0$. It follows that, if λ is large enough and in the region of the lemma,

$$\psi(x,\lambda) = B(\lambda)y(x,\lambda).$$

Since $W(\phi,\psi) = 1$, $\qquad B(\lambda)W(\phi,y) = 1.$

Now for a fixed λ, $x \to \infty$, (9.4.1) (with $A = 1$, $B = 0$) gives

$$\eta(x) \sim e^{i\xi(x)}\left\{1+\frac{1}{2i}\int_0^\infty e^{-i\xi(t)}R(t)\eta(t)\,dt\right\} = K(\lambda)e^{i\xi(x)}$$

say. As in previous cases, we also have

$$\eta'(x) \sim iK(\lambda)\{\lambda-q(x)\}^{\frac{1}{2}}e^{i\xi(x)}.$$

Hence

$$\begin{aligned} W(\phi,y) &= \{\lambda-q(x)\}^{-\frac{1}{4}}W[\{\lambda-q(x)\}^{\frac{1}{4}}\phi(x,\lambda),\, \eta(x,\lambda)] \\ &\sim \{\lambda-q(x)\}^{-\frac{1}{4}}[M(\lambda)e^{-i\xi(x)}iK(\lambda)\{\lambda-q(x)\}^{\frac{1}{4}}e^{i\xi(x)}+ \\ &\qquad +iM(\lambda)\{\lambda-q(x)\}^{\frac{1}{4}}e^{-i\xi(x)}K(\lambda)e^{i\xi(x)}] \\ &= 2iM(\lambda)K(\lambda). \end{aligned}$$

Hence $$2iB(\lambda)M(\lambda)K(\lambda) = 1,$$

i.e. $$\psi(x,\lambda) = \frac{y(x,\lambda)}{2iM(\lambda)K(\lambda)}. \tag{9.5.1}$$

Clearly $K(\lambda) \to 1$ as $\lambda \to \infty$ in the above region.

For a fixed x, and $\lambda \to \infty$ in the above region, this gives (if $\sin\alpha \neq 0$)

$$\psi(x,\lambda) \sim \frac{y(x,\lambda)}{i\lambda^{\frac{1}{2}}\sin\alpha} \sim \frac{e^{i\xi(x,\lambda)}}{i\lambda^{\frac{1}{4}}\{\lambda - q(x)\}^{\frac{1}{4}}\sin\alpha}.$$

Since

$$\xi(x) = \int_0^x \{\lambda - q(t)\}^{\frac{1}{2}}\,dt$$

$$= \int_0^x \{\lambda^{\frac{1}{2}} + O(\lambda^{-\frac{1}{2}})\}\,dt$$

$$= x\lambda^{\frac{1}{2}} + O(\lambda^{-\frac{1}{2}}),$$

this also gives $$\psi(x,\lambda) \sim \frac{e^{ix\sqrt{\lambda}}}{i\lambda^{\frac{1}{2}}\sin\alpha}. \tag{9.5.2}$$

Similarly $$\psi'(x,\lambda) \sim \frac{e^{ix\sqrt{\lambda}}}{\sin\alpha}. \tag{9.5.3}$$

If $\sin\alpha = 0$, the corresponding results are

$$\psi(x,\lambda) \sim \frac{e^{ix\sqrt{\lambda}}}{\cos\alpha}, \qquad \psi'(x,\lambda) = \frac{i\lambda^{\frac{1}{2}}e^{ix\sqrt{\lambda}}}{\cos\alpha}. \tag{9.5.4}$$

9.6. All the above results apply to regions which do not come too close to the positive real axis in the λ-plane. In the neighbourhood of the positive real axis the expression (9.1.3) has to be used, and for this purpose some further information about the functions $\psi_n(x)$ is needed.

We have $$\psi_n''(x) + \{\lambda_n - q(x)\}\psi_n(x) = 0.$$

Let $q(p_n) = \lambda_n$. Then for $0 < x < p_n$ the coefficient of $\psi_n(x)$ is positive. Hence $\psi_n(x)$ is concave downwards where it is positive, and upwards where it is negative; it therefore has just one maximum or minimum between consecutive zeros.

LEMMA 9.6 (*a*). *The successive maxima of $|\psi_n(x)|$ in $0 < x < p_n$ form a non-decreasing sequence.*

Let $$F(x) = \psi_n^2(x) + \frac{\psi'^2(x)}{\lambda_n - q(x)}.$$

Then $$F'(x) = \frac{\psi_n'^2(x)q'(x)}{\{\lambda_n - q(x)\}^2} \geqslant 0.$$

Hence $F(x)$ is non-decreasing. At a critical value of $\psi_n(x)$, $\psi_n'(x) = 0$, so that $F(x) = \psi_n^2(x)$. The lemma therefore follows.

LEMMA 9.6 (*b*). *Let $\eta_1, \eta_2,...$ be the zeros of $\psi_n(x)$. Then*

$$\frac{|\psi_n'(\eta_\nu)|}{|\lambda_n - q(\eta_\nu)|^{\frac{1}{2}}}$$

is a non-decreasing function of ν.

Since $$F(x) = \frac{\psi_n'^2(x)}{\lambda_n - q(x)}$$

at $x = \eta_1, \eta_2,...,$ this also follows from the above argument.

9.7. LEMMA 9.7. *For any fixed x (or x in a fixed interval)*

$$\psi_n'(x) = O(\lambda_n^{\frac{1}{2}} p_n^{-\frac{1}{2}}).$$

Let the successive maxima (or minima) and zeros of $\psi_n(x)$ in the interval $0 < x \leqslant \frac{1}{2}p_n$ be

$$\xi_1 < \eta_1 < \xi_2 < ... < \eta_{n'}.$$

Consider a particular η_μ. Then by Lemma 9.6 (*b*)

$$\frac{|\psi_n'(\eta_\mu)|}{|\lambda_n - q(\eta_\mu)|^{\frac{1}{2}}} \leqslant \frac{1}{n'-\mu+1} \sum_{\nu=\mu}^{n'} \frac{|\psi_n'(\eta_\nu)|}{|\lambda_n - q(\eta_\nu)|^{\frac{1}{2}}}.$$

Since $q(x)$ is convex downwards, $q(\frac{1}{2}x) \leqslant \frac{1}{2}q(x)$. Hence

$$q(\tfrac{1}{2}p_n) \leqslant \tfrac{1}{2}q(p_n) = \tfrac{1}{2}\lambda_n.$$

Hence $$|\psi_n'(\eta_\mu)| \leqslant \frac{\sqrt{2}}{n'-\mu+1} \sum_{\nu=\mu}^{n'} |\psi_n'(\eta_\nu)|. \tag{9.7.1}$$

Suppose e.g. that $\psi_n(x)$ is positive between ξ_ν and η_ν. Then

$$\int_{\xi_\nu}^{\eta_\nu} \psi_n(x)\, dx = \int_{\xi_\nu}^{\eta_\nu} \frac{-\psi_n''(x)}{\lambda_n - q(x)}\, dx$$

$$\geqslant \frac{1}{\lambda_n} \int_{\xi_\nu}^{\eta_\nu} \{-\psi_n''(x)\}\, dx$$

$$= \frac{1}{\lambda_n}\{\psi_n'(\xi_\nu) - \psi_n'(\eta_\nu)\}$$

$$= -\frac{\psi_n'(\eta_\nu)}{\lambda_n}.$$

Hence
$$|\psi_n'(\eta_\nu)| \leqslant \lambda_n \int\limits_{\xi_\nu}^{\eta_\nu} |\psi_n(x)|\,dx.$$

Hence (9.7.1) gives
$$|\psi_n'(\eta_\mu)| \leqslant \frac{\sqrt{2}\,\lambda_n}{n'-\mu+1} \int\limits_0^{\frac{1}{2}p_n} |\psi_n(x)|\,dx$$
$$\leqslant \frac{\sqrt{2}\,\lambda_n}{n'-\mu+1}\left[\int\limits_0^{\frac{1}{2}p_n} \{\psi_n(x)\}^2\,dx \int\limits_0^{\frac{1}{2}p_n} dx\right]^{\frac{1}{2}}$$
$$\leqslant \frac{\lambda_n p_n^{\frac{1}{2}}}{n'-\mu+1}. \tag{9.7.2}$$

Now let x be a given positive number. Let ξ_μ, $\xi_{\mu+1}$ be the nearest zeros of $\psi_n'(y)$ below and above x, η_μ the zero of $\psi_n(y)$ between ξ_μ and $\xi_{\mu+1}$. Then $|\psi_n'(x)| \leqslant |\psi_n'(\eta_\mu)|$. Now it is clear from the argument of § 7.2 that
$$n'-\mu \geqslant \frac{1}{\pi}\int\limits_x^{\frac{1}{2}p_n} \{\lambda_n - q(y)\}^{\frac{1}{2}}\,dy - \tfrac{3}{2}$$
$$\geqslant \frac{1}{\pi}\int\limits_x^{\frac{1}{2}p_n} (\tfrac{1}{2}\lambda_n)^{\frac{1}{2}}\,dy - \tfrac{3}{2}$$
$$> Ap_n \lambda_n^{\frac{1}{2}} \tag{9.7.3}$$
for n large enough. Hence
$$|\psi_n'(\eta_\mu)| < A\lambda_n^{\frac{1}{2}} p_n^{-\frac{1}{2}}, \tag{9.7.4}$$
and the lemma follows.

9.8. Lemma 9.8. *For a fixed* x
$$\psi_n(x) = O(p_n^{-\frac{1}{2}}).$$

Consider again an interval (ξ_ν, η_ν) where $\psi_n(x) > 0$, $\psi_n'(x) < 0$, $x \leqslant \frac{1}{2}p_n$. Then
$$\psi_n'(x)\psi_n''(x) = \{\lambda_n - q(x)\}\psi_n(x)\{-\psi_n'(x)\}$$
$$\geqslant \tfrac{1}{2}\lambda_n \psi_n(x)\{-\psi_n'(x)\}.$$

Hence
$$\int\limits_{\xi_\nu}^{\eta_\nu} \psi_n'(x)\psi_n''(x)\,dx \geqslant \tfrac{1}{2}\lambda_n \int\limits_{\xi_\nu}^{\eta_\nu} \psi_n(x)\{-\psi_n'(x)\}\,dx,$$
$$\psi_n'^2(\eta_\nu) - \psi_n'^2(\xi_\nu) \geqslant \tfrac{1}{2}\lambda_n\{\psi_n^2(\xi_\nu) - \psi_n^2(\eta_\nu)\},$$
$$\psi_n'^2(\eta_\nu) \geqslant \tfrac{1}{2}\lambda_n \psi_n^2(\xi_\nu).$$

Hence by (9.7.4), if $\eta_{\mu-1} < x < \eta_\mu$,

$$|\psi_n(\xi_\mu)| < Ap_n^{-\frac{1}{2}}$$

and the lemma follows.

It is easily verified that, in particular cases, these results are the best possible. For example, in the Hermite case, $q(x) = x^2$, $\lambda_n = 2n+1$, $p_n = (2n+1)^{\frac{1}{2}}$, so that Lemma 9.8 gives $\psi_n(x) = O(n^{-\frac{1}{4}})$. Actually

$$\psi_n(x) = \frac{e^{-\frac{1}{2}x^2}H_n(x)}{(n!\,2^n\pi^{\frac{1}{2}})^{\frac{1}{2}}}.$$

Now (Szegö, p. 94) for a fixed x

$$H_n(x) = O\left\{\frac{n!}{\Gamma(\frac{1}{2}n+1)}\right\}.$$

Hence

$$\psi_n(x) = O\left\{\frac{(n!)^{\frac{1}{2}}}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)}\right\} = O\left\{\frac{(n^{n+\frac{1}{2}}e^{-n})^{\frac{1}{2}}}{2^{\frac{1}{2}n}(\frac{1}{2}n)^{\frac{1}{2}n+\frac{1}{2}}e^{-\frac{1}{2}n}}\right\}$$
$$= O(n^{-\frac{1}{4}}).$$

9.9. We can now prove

THEOREM 9.9. *Let $f(y)$ belong to $L^2(0,\infty)$, and let $f(y) = 0$ in an interval of which x is an interior point. Then*

$$\sum_{n=0}^{\infty} c_n\psi_n(x) = 0.$$

This shows that, as in the case of ordinary Fourier series, the convergence of the series $\sum c_n\psi_n(x)$ depends only on the behaviour of the function $f(y)$ in the immediate neighbourhood of the point x. Of course we have also assumed that $f(y)$ is L^2, a condition not required in the theory of Fourier series.

Let $f(y) = 0$ for $x-\delta \leqslant y \leqslant x+\delta$.

We have to prove that, under the conditions of the theorem,

$$\lim \int_{|\lambda|=R} \Phi(x,\lambda)\,d\lambda = 0$$

as $R \to \infty$ through values not equal to any of the eigenvalues. We have

$$\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\,d\lambda = \sum_{n=0}^{\infty} c_n\psi_n(x)\int_{Re^{-i\eta}}^{Re^{i\eta}} \frac{d\lambda}{\lambda-\lambda_n} = \sum_{n=0}^{\infty} c_n\psi_n(x)I_n, \tag{9.9.1}$$

say. Now $\sum c_n^2$ is convergent, since $f(x)$ is $L^2(0,\infty)$. Let

$$\epsilon(R) = \Big(\sum_{\lambda_n > R} c_n^2\Big)^{\frac{1}{2}}.$$

Then

$$\left|\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\,d\lambda\right| \leqslant \sum_{\lambda_n \leqslant \frac{1}{2}R} |c_n \psi_n(x) I_n| + \sum_{\lambda_n > \frac{1}{2}R} |c_n \psi_n(x) I_n|$$

$$\leqslant \Big\{\sum_{\lambda_n \leqslant \frac{1}{2}R} c_n^2 \sum_{\lambda_n \leqslant \frac{1}{2}R} |\psi_n(x) I_n|^2\Big\}^{\frac{1}{2}} + \Big\{\sum_{\lambda_n > \frac{1}{2}R} c_n^2 \sum_{\lambda_n > \frac{1}{2}R} |\psi_n(x) I_n|^2\Big\}^{\frac{1}{2}}$$

$$= O\Big\{\sum_{\lambda_n \leqslant \frac{1}{2}R} |\psi_n(x) I_n|^2\Big\}^{\frac{1}{2}} + O\Big[\epsilon(\tfrac{1}{2}R)\Big\{\sum_{\lambda_n > \frac{1}{2}R} |\psi_n(x) I_n|^2\Big\}^{\frac{1}{2}}\Big].$$

Now I_n is bounded for all values of R and n (e.g. since we can take it along an arc of a circle with centre at $\lambda = \lambda_n$). Since

$$|\lambda - \lambda_n| \geqslant R - \lambda_n,$$

and the length of the contour is $2R\eta$, it is also

$$O\Big(\frac{R\eta}{|R-\lambda_n|}\Big).$$

From this and Lemma 9.8, it follows that

$$\sum_{\lambda_n \leqslant \frac{1}{2}R} |\psi_n(x) I_n|^2 = O\Big(\sum_{\lambda_n \leqslant \frac{1}{2}R} \frac{\eta^2}{p_n}\Big) = O(S_1),$$

say, and

$$\sum_{\lambda_n < \frac{1}{2}R} |\psi_n(x) I_n|^2 = O\Big\{\sum_{\frac{1}{2}R < \lambda_n \leqslant R - R\eta} \frac{R^2\eta^2}{(R-\lambda_n)^2 p_n}\Big\} +$$

$$+ O\Big\{\sum_{R-R\eta < \lambda_n \leqslant R+R\eta} \frac{1}{p_n}\Big\} + O\Big\{\sum_{\lambda_n < R+R\eta} \frac{R^2\eta^2}{(\lambda_n - R)^2 p_n}\Big\}$$

$$= O(S_2) + O(S_3) + O(S_4),$$

say.

In S_3, $p_n \geqslant p(R-R\eta)$, and the number of terms in the sum is $N(R+R\eta) - N(R-R\eta)$. Hence

$$S_3 \leqslant \frac{N(R+R\eta)-N(R-R\eta)}{p(R-R\eta)} = O\Big(\frac{R\eta}{R^{\frac{1}{2}}}\Big) = O(R^{\frac{1}{2}}\eta)$$

by (7.6.2), provided that

$$\sqrt{(R-R\eta)} \leqslant 2R\eta \leqslant R - R\eta,$$

which is true if

$$\frac{1}{2\sqrt{R}} \leqslant \eta \leqslant \tfrac{1}{3}.$$

Next

$$S_4 = \sum_{k=1}^{\infty} \sum_{R+2^{k-1}R\eta<\lambda_n\leqslant R+2^kR\eta} \frac{R^2\eta^2}{(\lambda_n-R)^2p_n}$$

$$\leqslant \sum_{k=1}^{\infty} \sum_{R+2^{k-1}R\eta<\lambda_n\leqslant R+2^kR\eta} \frac{1}{2^{2k-2}p_n}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{2^{2k-2}} \frac{N(R+2^kR\eta)-N(R+2^{k-1}R\eta)}{p(R+2^{k-1}R\eta)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{2k-2}} O\left(\frac{2^{k-1}R\eta}{R^{\frac{1}{2}}}\right) = O(R^{\frac{1}{2}}\eta).$$

Similarly $$S_2 = O(R^{\frac{1}{2}}\eta).$$

Lastly a similar argument gives

$$\sum_{2^{-k-1}R<\lambda\leqslant 2^{-k}R} \frac{1}{p_n} = O\left\{\frac{2^{-k-1}R}{(2^{-k-1}R)^{\frac{1}{2}}}\right\} = O\left(\frac{R^{\frac{1}{2}}}{2^{\frac{1}{2}k}}\right),$$

and summing terms of this type,

$$S_1 = O(R^{\frac{1}{2}}\eta^2).$$

Altogether $$\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\,d\lambda = O(R^{\frac{1}{2}}\eta)+O\{\epsilon(\tfrac{1}{2}R)R^{\frac{1}{2}}\eta^{\frac{1}{2}}\}. \tag{9.9.2}$$

This tends to zero if η is a suitable function of R, e.g. if

$$\eta = \frac{1}{R^{\frac{1}{2}}\{\epsilon(\frac{1}{2}R)+R^{-\frac{1}{2}}\}} \tag{9.9.3}$$

(or of course for smaller η, but later we want $R^{\frac{1}{2}}\eta \to \infty$).

9.10. Next consider the order of $\Phi(x,\lambda)$ on the parabola $v^2 = u$. We have

$$|\Phi(x,\lambda)| \leqslant \left\{\sum_{n=0}^{\infty} c_n^2 \sum_{n=0}^{\infty} \left|\frac{\psi_n(x)}{\lambda-\lambda_n}\right|^2\right\}^{\frac{1}{2}}$$

$$= O\left\{\sum_{n=0}^{\infty} \frac{p_n^{-1}}{(\lambda_n-u)^2+v^2}\right\}^{\frac{1}{2}}. \tag{9.10.1}$$

The last sum does not exceed

$$\sum_{\lambda_n\leqslant\frac{1}{2}u} \frac{1}{u^2p_n} + \sum_{\frac{1}{2}u<\lambda_n\leqslant u-v} \frac{1}{(u-\lambda_n)^2p_n} + \sum_{u-v<\lambda_n\leqslant u+v} \frac{1}{v^2p_n} +$$

$$+ \sum_{u+v<\lambda_n} \frac{1}{(\lambda_n-u)^2p_n}.$$

We consider these sums in the same way as those of the previous

section, with R and $R\eta$ replaced by u and v respectively. The third sum, for example, is

$$O\left\{\frac{1}{v^2}\frac{N(u+v)-N(u-v)}{p(u-v)}\right\} = O\left\{\frac{1}{v^2}\frac{v}{u^{\frac{1}{2}}}\right\} = O\left(\frac{1}{u}\right),$$

and similarly for the others. Hence on the parabola

$$\Phi(x,\lambda) = O(u^{-\frac{1}{2}}) = O(|\lambda|^{-\frac{1}{2}}). \tag{9.10.2}$$

This also holds throughout the part of the first quadrant above the parabola, since (9.10.1) is a decreasing function of v, for a fixed u. In the second quadrant

$$\Phi(x,\lambda) = O\left\{\sum_{n=0}^{\infty}\frac{p_n^{-1}}{(\lambda_n+|u|)^2}\right\}^{\frac{1}{2}} = O(1),$$

the convergence of the series following from the above argument.

Now we also have

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^{x-\delta}\phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int_{x+\delta}^{\infty}\psi(y,\lambda)f(y)\,dy.$$

Combining (9.3.2), (9.3.3), (9.3.5), (9.3.6),

$$\phi(x,\lambda) = O\{|\lambda^{-\frac{1}{4}}M(\lambda)e^{-ix\sqrt{\lambda}}|\}$$

for a fixed x, or x in a finite range. By (9.5.1) and (9.4.2)

$$\psi(x,\lambda) = O\{|\lambda^{-\frac{1}{4}}e^{i\xi(x)}/M(\lambda)|\}$$

for all x, and λ in the region allowed by § 9.2. Let λ be real and negative, say $\lambda = -\mu$. Then

$$\mathbf{I}\{\xi(x)\} = \int_0^x \{\mu+q(t)\}^{\frac{1}{2}}\,dt \geqslant x\sqrt{\mu}.$$

Hence

$$\Phi(x,-\mu) = O\left\{\frac{e^{-x\sqrt{\mu}}}{\mu^{\frac{1}{2}}}\int_0^{x-\delta} e^{y\sqrt{\mu}}|f(y)|\,dy\right\}+$$
$$+O\left\{\frac{e^{x\sqrt{\mu}}}{\mu^{\frac{1}{2}}}\int_{x+\delta}^{\infty} e^{-y\sqrt{\mu}}|f(y)|\,dy\right\}.$$

The first term is clearly $O(\mu^{-\frac{1}{2}}e^{-\delta\sqrt{\mu}})$; and so is the second, since the integral does not exceed

$$\left\{\int_{x+\delta}^{\infty} e^{-2y\sqrt{\mu}}\,dy\int_{x+\delta}^{\infty}|f(y)|^2\,dy\right\}^{\frac{1}{2}} \leqslant \left\{\frac{e^{-2(x+\delta)\sqrt{\mu}}}{2\sqrt{\mu}}\int_0^{\infty}|f(y)|^2\,dy\right\}^{\frac{1}{2}}.$$

Thus
$$\Phi(x,-\mu) = O(\mu^{-\frac{1}{2}}e^{-\delta\sqrt{\mu}}). \tag{9.10.3}$$

Let
$$F(\lambda) = \lambda^{\frac{1}{2}}e^{-i\delta\lambda^{\frac{1}{2}}}\Phi(x,\lambda),$$

where $\lambda^{\frac{1}{2}}$ is real on the positive real axis. By (9.10.3), $F(\lambda)$ is bounded on the negative real axis. On the parabola $v^2 = u$

$$e^{-i\delta\lambda^{\frac{1}{2}}} = \exp\left\{-i\delta u^{\frac{1}{2}}+\frac{\delta v}{2u^{\frac{1}{2}}}+\dots\right\} = O(1).$$

Hence, by (9.10.2), $F(y)$ is bounded on the parabola. It is therefore bounded throughout the region between the parabola and the negative real axis, by the Phragmén-Lindelöf theorem. Hence, putting $\lambda = re^{i\theta}$,

$$\Phi(x,\lambda) = O(r^{-\frac{1}{2}}e^{-\delta r^{\frac{1}{2}}\sin\frac{1}{2}\theta}) = O(r^{-\frac{1}{2}}e^{-\delta r^{\frac{1}{2}}\theta/\pi}).$$

Hence, if $0 < \eta < \frac{1}{2}\pi$, $R\sin^2\eta \geqslant \cos\eta$,

$$\int\limits_{|\lambda|=R,\,\eta\leqslant\theta\leqslant\pi} \Phi(x,\lambda)\,d\lambda = O\left(\int\limits_{\eta}^{\pi} R^{\frac{1}{2}}e^{-\delta R^{\frac{1}{2}}\theta/\pi}\,d\theta\right) = O(e^{-\delta R^{\frac{1}{2}}\eta/\pi}). \tag{9.10.4}$$

The same result holds, of course, for the integral over $-\pi \leqslant \theta \leqslant -\eta$. Theorem 9.9 now follows from (9.9.2) and (9.10.4).

9.11. In view of the above theorem, we can now confine our attention to a finite interval. We first prove

LEMMA 9.11. *Let $f(x)$ be of bounded variation over a finite interval (a,b), and zero elsewhere. Then*

$$c_n\psi_n(x) = O\left(\frac{1}{n}\right).$$

Let $f(x) = f_1(x)-f_2(x)$, where f_1 and f_2 are bounded and non-increasing in (a,b). Then

$$c_n = \int\limits_a^b \psi_n(y)f_1(y)\,dy - \int\limits_a^b \psi_n(y)f_2(y)\,dy.$$

By the second mean-value theorem

$$\begin{aligned}\int\limits_a^b \psi_n(y)f_1(y)\,dy &= f_1(a+0)\int\limits_a^\beta \psi_n(y)\,dy \quad (a<\beta<b)\\ &= -f_1(a+0)\int\limits_a^\beta \frac{\psi_n''(y)}{\lambda_n-q(y)}\,dy\\ &= -\frac{f_1(a+0)}{\lambda_n-q(\beta)}\int\limits_\alpha^\beta \psi_n''(y)\,dy \quad (a<\alpha<\beta)\\ &= \frac{f_1(a+0)}{\lambda_n-q(\beta)}\{\psi_n'(\alpha)-\psi_n'(\beta)\}\\ &= O(\lambda_n^{-\frac{1}{2}}p_n^{-\frac{1}{2}})\end{aligned}$$

by Lemma 9.7. Similarly for the other integral. Hence by Lemma 9.8

$$c_n \psi_n(x) = O(\lambda_n^{-\frac{1}{4}} p_n^{-1}).$$

By Theorem (7.3)

$$n = O(\lambda_n^{\frac{1}{4}} p_n),$$

$$\lambda_n^{-\frac{1}{4}} p_n^{-1} = O(n^{-1})$$

and the lemma follows.

9.12. Theorem 9.12. *Let $q(x)$ satisfy the conditions stated in § 9.2, and let $f(y)$ be of bounded variation over an interval including the point x, and zero outside this interval. Then*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

We start again from (9.9.1), but now use $c_n \psi_n(x) = O(1/n)$ instead of $\sum c_n^2 < \infty$. We obtain

$$\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\,d\lambda = O\Big\{\sum_{\lambda_n \leqslant \frac{1}{2}R} \frac{\eta}{n}\Big\} + O\Big\{\sum_{\frac{1}{2}R<\lambda_n \leqslant R-R\eta} \frac{R\eta}{n(R-\lambda_n)}\Big\} +$$

$$+O\Big\{\sum_{R-R\eta<\lambda_n\leqslant R+R\eta} \frac{1}{n}\Big\} + O\Big\{\sum_{\lambda_n>R+R\eta} \frac{R\eta}{n(\lambda_n-R)}\Big\}$$

$$= O(\textstyle\sum_1)+O(\sum_2)+O(\sum_3)+O(\sum_4),$$

say. Now

$$\sum\nolimits_3 = O\left\{\frac{N(R+R\eta)-N(R-R\eta)}{N(R-R\eta)}\right\} = O(\eta)$$

by (7.6.3), provided that $\eta \geqslant R^{-\frac{1}{2}}$. Next

$$\sum\nolimits_4 = \sum_{k=1}^{\infty} \sum_{R+2^{k-1}R\eta<\lambda_n \leqslant R+2^k R\eta} \frac{R\eta}{n(\lambda_n-R)}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{N(R+2^k R\eta)-N(R+2^{k-1}R\eta)}{N(R+2^{k-1}R\eta)} \frac{1}{2^{k-1}}$$

$$= \sum_{k=1}^{\infty} O\left(\frac{2^{k-1}R\eta}{R+2^{k-1}R\eta}\,\frac{1}{2^{k-1}}\right)$$

$$= \sum_{2^{k-1}\eta\leqslant 1} O(\eta) + \sum_{2^{k-1}\eta>1} O\left(\frac{1}{2^{k-1}}\right)$$

$$= O\left(\eta \log\frac{1}{\eta}\right) + O(\eta) = O\left(\eta\log\frac{1}{\eta}\right).$$

Similarly for $\sum_2$; and

$$\sum_{\frac{1}{4}R<\lambda_n\leqslant\frac{1}{2}R} \frac{\eta}{n} = O\left\{\eta\frac{N(\frac{1}{2}R)-N(\frac{1}{4}R)}{N(\frac{1}{4}R)}\right\} = O(\eta)$$

by (7.6.3), and adding $O(\log R)$ terms of this form,

$$\sum\nolimits_1 = O(\eta \log R).$$

We may take for example $\eta = R^{-\alpha}$, where $0 < \alpha < \frac{1}{6}$. Then

$$\lim_{R\to\infty} \int\limits_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\, d\lambda = 0.$$

Now $\lambda = R\exp(iR^{-\alpha}) \sim R+iR^{1-\alpha}$ lies in the region specified in Lemma 9.2, if α is in the above interval. Hence on the remaining arc of the circle $|\lambda| = R$ we may apply the asymptotic formulae obtained in §§ 9.3–9.5.

Now

$$\Phi(x,\lambda) = \psi(x,\lambda) \int\limits_{x-\delta}^{x} \phi(y,\lambda)f(y)\, dy + \phi(x,\lambda) \int\limits_{x}^{x+\delta} \psi(y,\lambda)f(y)\, dy.$$

The second term can be written in the form

$$f(x+0)\phi(x,\lambda) \int\limits_{x}^{x+\delta} \psi(y,\lambda)\, dy + \phi(x,\lambda) \int\limits_{x}^{x+\delta} \psi(y,\lambda)\{f(y)-f(x+0)\}\, dy$$
$$= \Phi_1+\Phi_2.$$

Now

$$\int\limits_{x}^{x+\delta} \psi(y,\lambda)\, dy = -\int\limits_{x}^{x+\delta} \frac{\psi''(y,\lambda)}{\lambda-q(y)}\, dy$$
$$= \frac{\psi'(x,\lambda)}{\lambda-q(x)} - \frac{\psi'(x+\delta,\lambda)}{\lambda-q(x+\delta)} + \int\limits_{x}^{x+\delta} \psi'(y,\lambda)\frac{q'(y)\, dy}{\{\lambda-q(y)\}^2}.$$

Suppose, for example, that $\sin\alpha \neq 0$. Using (9.3.2) and (9.5.3),

$$\Phi_1 = f(x+0)\cos(x\sqrt{\lambda})\{\sin\alpha+O(|\lambda|^{-\frac{1}{2}})\}\times$$
$$\times\left\{\frac{e^{ix\lambda^{\frac{1}{2}}}\{1+o(1)\}}{\{\lambda-q(x)\}\sin\alpha} + O\left(\frac{|e^{i(x+\delta)\lambda^{\frac{1}{2}}}|}{|\lambda|}\right) + O\left(\frac{|e^{ix\lambda^{\frac{1}{2}}}|}{|\lambda|^2}\right)\right\}$$
$$\sim f(x+0)\frac{\cos(x\sqrt{\lambda})e^{ix\sqrt{\lambda}}}{\lambda} = f(x+0)\left\{\frac{1}{2\lambda} + O\left(\frac{|e^{2ix\sqrt{\lambda}}|}{\lambda}\right)\right\}.$$

Hence $$\lim_{R\to\infty} \int\limits_{|\arg\lambda|\geqslant\eta} \Phi_1(x,\lambda)\, d\lambda = \tfrac{1}{2}f(x+0).$$

In Φ_2, we can write

$$f(y)-f(x+0) = g_1(y)-g_2(y),$$

where g_1 and g_2 are non-decreasing and tend to 0 as $y \to x$. Now

$$\int\limits_x^{x+\delta} \mathbf{R}\{\psi(y,\lambda)\}g_1(y)\,dy = g_1(x+\delta-0)\int\limits_{x'}^{x+\delta} \mathbf{R}\psi(y,\lambda)\,dy$$

by the second mean-value theorem; and as before this integral is $O\{|e^{ix\sqrt{\lambda}}/\lambda|\}$. Similarly for the imaginary part, and for the corresponding integrals involving g_2. Altogether we obtain

$$\Phi_2 = O\left(\frac{g_1(x+\delta)+g_2(x+\delta)}{|\lambda|}\right).$$

Hence
$$\int\limits_{|\arg\lambda|>\eta} \Phi(x,\lambda)\,d\lambda = O\{g_1(x+\delta)+g_2(x+\delta)\}.$$

Since δ can be taken arbitrarily small, this contributes 0 to the final result.

Similarly for the other part of $\Phi(x,\lambda)$, and the theorem follows.

9.13. Combining Theorem 9.9 and Theorem 9.12, the final result of the analysis of this chapter is

THEOREM 9.13. *Let $f(y)$ belong to $L^2(0,\infty)$, and let $f(y)$ be of bounded variation over an interval including the point x. Let $q(x)$ satisfy the conditions stated in* § 9.2. *Then*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \sum_{n=0}^{\infty} c_n\,\psi_n(x).$$

REFERENCE

Titchmarsh (10).

X

SUMMABILITY OF THE SERIES EXPANSION

10.1. It will be assumed here merely that $q(x)$ is continuous and tends steadily to infinity, but not that it satisfies the other conditions of Chapter IX. The asymptotic formulae previously used are no longer available, or at any rate they have not been proved under such general conditions. Consequently it is no longer possible to prove the convergence of the eigenfunction expansion under Fourier conditions. We can, however, prove the following summability theorem.

THEOREM 10.1. *Let $q(x)$ be continuous and tend steadily to infinity, and let $f(x)$ be $L^2(0,\infty)$. Then*

$$\lim_{\nu\to\infty}\sum_{n=0}^{\infty}\frac{\nu}{\nu+\lambda_n}c_n\psi_n(x)=f(x) \tag{10.1.1}$$

for every value of x for which

$$\chi(\eta)=\int_0^{\eta}|f(x+y)-f(x)|\,dy=o(\eta) \tag{10.1.2}$$

as $\eta\to 0$. In particular (i) *it holds almost everywhere, and* (ii) *it holds wherever $f(x)$ is continuous.*

In view of (9.1.3), this is a question of the behaviour of $\Phi(x,\lambda)$ on the negative real axis. Putting $\lambda=-t^2$, it becomes a question of the solutions of

$$\frac{d^2y}{dx^2}=\{t^2+q(x)\}y$$

for real t, and $q(x)$ tending steadily to infinity. This is dealt with in the following lemmas.

10.2. LEMMA 10.2. *Let $Q(x)>0$ for $x\geqslant x_0$, and let $y(x)$ be a solution of*

$$\frac{d^2y}{dx^2}=Q(x)y$$

belonging to $L^2(x_0,\infty)$. Then $y(x)$ and $y'(x)$ have opposite signs, and tend steadily to zero as $x\to\infty$.

Suppose, for example, that $y(x_0)>0$. Then $y'(x_0)<0$, or $y(x)$ would tend steadily to infinity, since it is convex downwards where

it is positive. Thus $y'(x) < 0$ as long as $y(x) > 0$. But $y(x)$ cannot change sign; for if x_1 were its first zero, and $y(x) < 0$ for x just greater than x_1, there would be a point $x_1+\delta$ such that $y(x_1+\delta) < 0$, $y'(x_1+\delta) < 0$. Hence $y(x)$ would decrease steadily past this point, which is impossible if it belongs to $L^2(x_0, \infty)$.

It follows that, for $x \geqslant x_0$, $y(x) > 0$ and $y'(x) < 0$. Hence $y(x)$ decreases steadily, and so tends to a limit, which must be zero if $y(x)$ is L^2. Since $y''(x) > 0$, $y'(x)$ increases steadily, and so tends to a limit, which clearly must also be zero.

10.3. Lemma 10.3. *In addition to the above conditions, let $Q(x)$ be non-decreasing. Then, if $y(x_0) > 0$,*

$$y(x) \leqslant y(x_0)\exp\left[-\int_{x_0}^{x} \{Q(u)\}^{\frac{1}{2}}\, du\right] \quad (x > x_0). \tag{10.3.1}$$

We have for $x \geqslant x_0$

$$-y'(x)y''(x) = Q(x)y(x)\{-y'(x)\} \geqslant Q(x_0)y(x)\{-y'(x)\}.$$

Integrating from x_0 to x_1,

$$\tfrac{1}{2}\{y'^2(x_0)-y'^2(x_1)\} \geqslant \tfrac{1}{2}Q(x_0)\{y^2(x_0)-y^2(x_1)\}.$$

Making $x_1 \to \infty$, and replacing x_0 by x, it follows that

$$y'^2(x) \geqslant Q(x)y^2(x).$$

Hence

$$-y'(x)/y(x) \geqslant \{Q(x)\}^{\frac{1}{2}}. \tag{10.3.2}$$

Integrating from x_0 to x,

$$\log y(x_0)-\log y(x) \geqslant \int_{x_0}^{x} \{Q(u)\}^{\frac{1}{2}}\, du,$$

and the lemma follows.

10.4. Lemma 10.4. *Let $Q(x)$ satisfy all the above conditions, take $x_0 = 0$, and let*

$$y(0)\cos\alpha+y'(0)\sin\alpha = 1, \tag{10.4.1}$$

where $\sin\alpha \neq 0$ (*instead of* $y(0)$ *being given*). *Then, if* $Q(0) > \operatorname{cosec}^2\alpha$,

$$|y(x)| \leqslant \frac{\exp\left[-\int_{0}^{x} \{Q(u)\}^{\frac{1}{2}}\, du\right]}{\{Q(0)\}^{\frac{1}{2}}|\sin\alpha|-1}. \tag{10.4.2}$$

By (10.4.1) and (10.3.2)

$$|y'(0)\sin\alpha| \leqslant 1+|y(0)| \leqslant 1+\{Q(0)\}^{-\frac{1}{2}}|y'(0)|,$$

$$|y'(0)| \leqslant \frac{1}{|\sin\alpha|-\{Q(0)\}^{-\frac{1}{2}}}.$$

Also by (10.3.1) and (10.3.2)

$$|y(x)| \leqslant |y(0)|\exp\left[-\int_0^x \{Q(u)\}^{\frac{1}{2}}\,du\right]$$
$$\leqslant |y'(0)|\{Q(0)\}^{-\frac{1}{2}}\exp\left[-\int_0^x \{Q(u)\}^{\frac{1}{2}}\,du\right],$$

and (10.4.2) follows.

10.5. Lemma 10.5. *Under the same conditions*

$$|y(x)| \geqslant \frac{2y(x_0)\exp[-(x-x_0)\{Q(x+1)\}^{\frac{1}{2}}]}{1+\{1+1/Q(x+1)\}^{\frac{1}{2}}}. \tag{10.5.1}$$

Suppose, for example, that $y(x)$ is positive. Since $Q(x)$ is non-decreasing,

$$-y'(x)y''(x) = Q(x)y(x)\{-y'(x)\} \leqslant Q(x_1)y(x)\{-y'(x)\} \quad (x \leqslant x_1).$$

Integrating from x to x_1

$$\tfrac{1}{2}\{y'^2(x)-y'^2(x_1)\} \leqslant \tfrac{1}{2}Q(x_1)\{y^2(x)-y^2(x_1)\}$$
$$\leqslant \tfrac{1}{2}Q(x_1)y^2(x).$$

Writing temporarily $a = \{Q(x_1)\}^{\frac{1}{2}}$, $b = -y'(x_1)$, this gives

$$\left(\frac{dy}{dx}\right)^2 \leqslant a^2y^2+b^2.$$

Hence
$$\int_y^{y_0} \frac{dy}{\sqrt{(a^2y^2+b^2)}} \leqslant x-x_0,$$

where $y_0 = y(x_0)$. Hence

$$\frac{1}{a}\log\frac{y_0+\sqrt{(y_0^2+b^2/a^2)}}{y+\sqrt{(y^2+b^2/a^2)}} \leqslant x-x_0,$$

$$y+\sqrt{\left(y^2+\frac{b^2}{a^2}\right)} \geqslant \left\{y_0+\sqrt{\left(y_0^2+\frac{b^2}{a^2}\right)}\right\}e^{-a(x-x_0)} \geqslant 2y_0\,e^{-a(x-x_0)}. \tag{10.5.2}$$

Also, since y is positive and convex downwards,

$$|y'(x)| \leqslant \frac{y(x')-y(x)}{x-x'} \quad (0 < x' < x).$$

Taking $x' = x-1$, this gives

$$|y'(x)| \leqslant y(x-1).$$

Hence, taking $x_1 = x+1$ in (10.5.2),

$$y(x)+\left\{y^2(x)+\frac{y^2(x)}{Q(x+1)}\right\}^{\frac{1}{2}} \geqslant 2y_0\exp[-\{Q(x+1)\}^{\frac{1}{2}}(x-x_0)],$$

and (10.5.1) follows.

10.6. LEMMA 10.6. *Let $Q(x) = t^2+q(x)$, where $q(x)$ is non-decreasing and independent of t, and let $y(x)$ satisfy the above conditions. Then as $t \to \infty$*

$$y(x) \sim y(0)e^{-xt} \tag{10.6.1}$$

uniformly in any fixed range of values of x, if $y(0)$ is given; while if (10.4.1) *holds,*

$$y(x) \sim -\frac{e^{-xt}}{t\sin\alpha}. \tag{10.6.2}$$

The first result follows at once from (10.3.1) and (10.5.1).

It follows from (10.3.2), with $x = 0$, that $|y'(0)/y(0)|$ tends to infinity with t, and hence, if (10.4.1) holds, that $y'(0)$ has the same sign as $\sin\alpha$, if t is large enough. Hence $y(x)$ has the opposite sign to $\sin\alpha$. Thus the sign in (10.6.2) is correct. It will then be sufficient to consider one case, e.g. that in which $y(x)$ is positive.

The required upper bound for $y(x)$ follows at once from (10.4.2). Also (10.4.1) gives

$$1 \leqslant |y(0)|+|y'(0)\sin\alpha| \leqslant \{Q(0)\}^{-\frac{1}{2}}|y'(0)|+|y'(0)\sin\alpha|$$

by (10.3.2) with $x = 0$. Hence

$$|y'(0)| \geqslant \frac{1}{\{Q(0)\}^{-\frac{1}{2}}+\sin\alpha}. \tag{10.6.3}$$

Also, if $x_1 > x$,

$$\begin{aligned} y(x_1)-y(x) &= \int_x^{x_1} y'(\xi)\,d\xi \\ &= [-y'(\xi)(x_1-\xi)]_x^{x_1}+\int_x^{x_1} y''(\xi)(x_1-\xi)\,d\xi \\ &= y'(x)(x_1-x)+\int_x^{x_1} Q(\xi)y(\xi)(x_1-\xi)\,d\xi, \end{aligned}$$

$$y'(x) = \frac{y(x_1)-y(x)}{x_1-x}-\frac{1}{x_1-x}\int_x^{x_1} Q(\xi)y(\xi)(x_1-\xi)\,d\xi,$$

and hence $\quad |y'(x)| \leqslant \dfrac{y(x)-y(x_1)}{x_1-x}+(x_1-x)Q(x_1)y(x).$

Now, by (10.5.1),

$$y(x_1) \geqslant y(x)e^{-(x_1-x)t}\{1+O(1/t)\}$$

if x and x_1 lie in a fixed interval. Hence

$$|y'(x)| \leqslant y(x)\left[\frac{1-e^{-(x_1-x)t}}{x_1-x}+O\left\{\frac{1}{(x_1-x)t}\right\}+O\{(x_1-x)Q(x_1)\}\right].$$

Taking $(x_1-x)t$ small and $(x_1-x)t^2$ large, e.g. $x_1 = x+t^{-\frac{3}{2}}$, we obtain

$$|y'(x)| \leqslant ty(x)(1+\delta) \quad (t > t_0(\delta)). \tag{10.6.4}$$

The lower bound for $y(x)$ implied by (10.6.2) now follows from (10.5.1), (10.6.4), and (10.6.3).

10.7. Proof of Theorem 10.1. Since $f(x)$ is $L^2(0,\infty)$, (9.1.3) holds, and the theorem is true if

$$\Phi(x,-t^2) \sim -f(x)/t^2 \tag{10.7.1}$$

as $t \to \infty$. Now

$$\Phi(x,-t^2) = \psi(x,-t^2)\int_0^x \phi(y,-t^2)f(y)\,dy + \phi(x,-t^2)\int_x^\infty \psi(y,-t^2)f(y)\,dy.$$

Since $\phi(0) = \sin\alpha$, $\phi'(0) = -\cos\alpha$, $\theta(0) = \cos\alpha$, $\theta'(0) = \sin\alpha$, and

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda),$$

we obtain

$$\psi(0,-t^2)\cos\alpha+\psi'(0,-t^2)\sin\alpha = 1. \tag{10.7.2}$$

Suppose first that $\sin\alpha \neq 0$. By (1.7.3)

$$\phi(x) = \cosh xt\sin\alpha+O(e^{xt}/t) \tag{10.7.3}$$

as $t \to \infty$, uniformly in any finite x-interval; and $y(x) = \psi(x,-t^2)$ satisfies (10.6.2), uniformly in any finite x-interval. Also by (10.4.2)

$$\psi(x,-t^2) = O(e^{-xt})$$

for all x and sufficiently large t. Hence, if $\delta > 0$,

$$\psi(x,-t^2)\int_0^{x-\delta} \phi(y,-t^2)f(y)\,dy = O\left\{\frac{e^{-xt}}{t}\int_0^{x-\delta} e^{yt}|f(y)|\,dy\right\} = O(e^{-\delta t})$$

and

$$\phi(x,-t^2)\int_{x+\delta}^\infty \psi(y,-t^2)f(y)\,dy = O\left\{e^{xt}\int_{x+\delta}^\infty e^{-yt}|f(y)|\,dy\right\}$$

$$= O\left[e^{xt}\left\{\int_{x+\delta}^\infty e^{-2yt}\,dy\int_{x+\delta}^\infty |f(y)|^2\,dy\right\}^{\frac{1}{2}}\right] = O(e^{-\delta t}).$$

The remainder of $\Phi(x,-t^2)$ can be written

$$f(x)\left\{\psi(x,-t^2)\int_{x-\delta}^{x}\phi(y,-t^2)\,dy+\phi(x,-t^2)\int_{x}^{x+\delta}\psi(y,-t^2)\,dy\right\}+$$
$$+\psi(x,-t^2)\int_{x-\delta}^{x}\phi(y,-t^2)\{f(y)-f(x)\}\,dy+$$
$$+\phi(x,-t^2)\int_{x}^{x+\delta}\psi(y,-t^2)\{f(y)-f(x)\}\,dy.$$

The coefficient of $f(x)$ is asymptotic to

$$-\frac{e^{-xt}}{t\sin\alpha}\int_{x-\delta}^{x}\tfrac{1}{2}e^{yt}\sin\alpha\,dy-\tfrac{1}{2}e^{xt}\sin\alpha\int_{x}^{x+\delta}\frac{e^{-yt}}{t\sin\alpha}\,dy=-\frac{1}{t^2}+O(e^{-\delta t}).$$

If (10.1.2) holds, the last term in the previous expression is

$$O\left\{e^{xt}\int_{x}^{x+\delta}\frac{e^{-yt}}{t}|f(y)-f(x)|\,dy\right\}$$
$$=O\left\{\frac{1}{t}\int_{0}^{\delta}e^{-ut}|f(x+u)-f(x)|\,du\right\}$$
$$=O\left\{\frac{1}{t}[e^{-ut}\chi(u)]_0^{\delta}+\int_{0}^{\delta}e^{-ut}\chi(u)\,du\right\}$$
$$=O(e^{-\delta t}/t)+o\left(\int_{0}^{\delta}e^{-ut}u\,du\right)=o\,(t^{-2}).$$

Similarly for the second term; altogether

$$\Phi(x,-t^2)=-f(x)/t^2+o\,(t^{-2}),$$

the required result. A similar argument holds in the case $\sin\alpha=0$, using (1.7.5) and (10.6.1).

10.8. An alternative proof of Theorem 9.12. There is a well-known argument in the ordinary theory of Fourier series by which we can deduce the convergence of the Fourier series of a function of bounded variation from its summability $(C,1)$. If $f(x)$ is of bounded variation over $(0,2\pi)$, and a_n, b_n are its Fourier coefficients, then

$$a_n=O\left(\frac{1}{n}\right),\qquad b_n=O\left(\frac{1}{n}\right).$$

The series
$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$
is summable $(C, 1)$ to the sum $f(x)$ for each x; hence, by Hardy's 'Tauberian' theorem, it also converges to $f(x)$.

We are now in possession of the materials for a corresponding proof of Theorem 9.12, viz. Lemma 9.11 and Theorem 10.1; but, instead of Hardy's Tauberian theorem, we have to use a case of Wiener's general Tauberian theorem.

Let $a_n = c_n\psi_n(x)$, so that $a_n = O(1/n)$ by Lemma 9.11. Let
$$S = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$
Then by Theorem 10.1
$$-\lambda\Phi(x,-\lambda) = \sum_{n=0}^{\infty}\frac{\lambda a_n}{\lambda+\lambda_n}\to S$$
as $\lambda\to\infty$. Let
$$S(x) = \sum_{\lambda_n\leqslant x}a_n.$$
Then
$$S(\lambda)+\lambda\Phi(x,-\lambda) = \sum_{\lambda_n\leqslant\lambda}a_n\left(1-\frac{\lambda}{\lambda+\lambda_n}\right)-\sum_{\lambda_n>\lambda}\frac{a_n\lambda}{\lambda+\lambda_n}$$
$$= O\left\{\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n(\lambda+\lambda_n)}\right\}+O\left\{\sum_{\lambda_n>\lambda}\frac{\lambda}{n(\lambda+\lambda_n)}\right\}$$
$$= O\left(\frac{1}{\lambda}\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n}\right)+O\left(\lambda\sum_{\lambda_n>\lambda}\frac{1}{n\lambda_n}\right).$$
Now
$$\sum_{\lambda<\lambda_n\leqslant 2\lambda}\frac{1}{n\lambda_n}\leqslant\frac{1}{\lambda}\sum_{\lambda<\lambda_n\leqslant 2\lambda}\frac{1}{n}=\frac{1}{\lambda}\sum_{N(\lambda)<n\leqslant N(2\lambda)}\frac{1}{n}$$
$$= O\left\{\frac{1}{\lambda}\,\frac{N(2\lambda)-N(\lambda)}{N(\lambda)}\right\} = O\left(\frac{1}{\lambda}\right)$$
by (7.6.3). Replacing λ by 2λ, 4λ,... and adding, we obtain
$$\sum_{\lambda_n>\lambda}\frac{1}{n\lambda_n} = O\left(\frac{1}{\lambda}\right).$$
Similarly
$$\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n} = O(\lambda).$$
Hence
$$S(\lambda)+\lambda\Phi(x,-\lambda) = O(1),$$
$$S(\lambda) = O(1).$$

Also $$-\lambda\Phi(x,-\lambda) = \int\limits_{-0}^{\infty} \frac{\lambda}{\lambda+u}\, dS(u) = \int\limits_{0}^{\infty} \frac{\lambda}{(\lambda+u)^2} S(u)\, du.$$

Hence

$$S = \lim_{\lambda\to\infty} \int\limits_{0}^{\infty} \frac{\lambda}{(\lambda+u)^2} S(u)\, du$$

$$= \lim_{\xi\to\infty} \int\limits_{-\infty}^{\infty} \frac{e^{\xi}}{(e^{\xi}+e^{\eta})^2} S(e^{\eta})e^{\eta}\, d\eta$$

$$= \lim_{\xi\to\infty} \int\limits_{-\infty}^{\infty} \frac{e^{\xi-\eta}}{(e^{\xi-\eta}+1)^2} S(e^{\eta})\, d\eta$$

$$= \lim_{\xi\to\infty} \int\limits_{-\infty}^{\infty} K_1(\xi-\eta)S(e^{\eta})\, d\eta,$$

where $$K_1(x) = \frac{e^x}{(e^x+1)^2}.$$

Now

$$\int\limits_{-\infty}^{\infty} K_1(x)e^{-iux}\, dx = \int\limits_{-\infty}^{\infty} \frac{e^{x-iux}}{(e^x+1)^2}\, dx = \int\limits_{0}^{\infty} \frac{t^{-iu}}{(t+1)^2}\, dt$$

$$= \Gamma(1+iu)\Gamma(1-iu) \neq 0,$$

and $$\int\limits_{-\infty}^{\infty} K_1(x)\, dx = 1.$$

Hence, by Wiener's Theorem 4,†

$$\lim_{\xi\to\infty} \int\limits_{-\infty}^{\infty} K_2(\xi-\eta)S(e^{\eta})\, d\eta = S\int\limits_{-\infty}^{\infty} K_2(x)\, dx,$$

where $$K_2(\xi) = 0 \quad (\xi < 0), \qquad e^{-\xi} \quad (\xi > 0).$$

Thus

$$S = \lim_{\xi\to\infty} \int\limits_{-\infty}^{\xi} e^{\eta-\xi}S(e^{\eta})\, d\eta$$

$$= \lim_{x\to\infty} \frac{1}{x} \int\limits_{0}^{x} S(y)\, dy.$$

† N. Wiener, *The Fourier Integral*, pp. 73–4.

Let δ be a number between 0 and 1, independent of x. Then

$$S = \frac{(1+\delta)S-S}{\delta} = \lim_{x\to\infty}\frac{1}{\delta x}\left\{\int\limits_0^{(1+\delta)x} S(y)\,dy - \int\limits_0^{x} S(y)\,dy\right\}$$

$$= \lim_{x\to\infty}\frac{1}{\delta x}\int\limits_x^{(1+\delta)x} S(y)\,dy$$

$$= \lim_{x\to\infty}\left[S(x)+\frac{1}{\delta x}\int\limits_x^{(1+\delta)x}\{S(y)-S(x)\}\,dy\right].$$

Now

$$S(y)-S(x) = \sum_{x<\lambda_n\leqslant y} a_n = \sum_{x<\lambda_n\leqslant y} O\left(\frac{1}{n}\right)$$

$$= O\left\{\sum_{x<\lambda_n\leqslant(1+\delta)x}\frac{1}{n}\right\} = O\left\{\sum_{N(x)<n\leqslant N\{(1+\delta)x\}}\frac{1}{n}\right\}$$

$$= O\left[\frac{N\{(1+\delta)x\}-N(x)}{N(x)}\right] = O(\delta)$$

by (7.6.3), provided that $x^{-\frac{1}{2}} \leqslant \delta$. Hence

$$\frac{1}{\delta x}\int\limits_x^{(1+\delta)x}\{S(y)-S(x)\}\,dy = \frac{1}{\delta x}\int\limits_x^{(1+\delta)x} O(\delta)\,dy$$

$$= O(\delta).$$

Since δ is arbitrary, it follows that

$$\lim_{x\to\infty} S(x) = S.$$

REFERENCES

Titchmarsh (9), (10).

REFERENCES

A general account of the subject up to 1920, with many references, is given by E. Hilb and O. Szász, *Encyklopädie der math. Wiss.* II. III. 2 (1923–7), 1244–66.

See also H. Burkhardt, *Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*, Jahresbericht der Deutschen Math.-Vereinigung, 10 (1908).

On the special case of orthogonal polynomials, see G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. xxiii (1939).

The general theory of linear transformations is given by M. H. Stone, *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. xv (1932).

Bell, R. P.

(1) Eigen-values and eigen-functions for the operator $d^2/dx^2-|x|$, *Phil. Mag.* 35 (1944), 582–8.

(2) The occurrence and properties of molecular vibrations with $V(x) = ax^4$, *Proc. Roy. Soc.* (A), 183 (1945), 328–37.

Birkhoff, G. D.

(1) On the asymptotic character of the solutions of certain linear differential equations containing a parameter, *Trans. Amer. Math. Soc.* 9 (1908), 219–31.

(2) Boundary value and expansion problems of ordinary linear differential equations, *Trans. Amer. Math. Soc.* 9 (1908), 373–95.

(3) Quantum mechanics and asymptotic series, *Bull. Amer. Math. Soc.* 39 (1933), 681–700.

Brillouin, L.

(1) La mécanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations successives, *Comptes rendus*, 183 (July 1926), 24–6.

Burnett, D.

(1) The distribution of velocities in a slightly non-uniform gas, *Proc. London Math. Soc.* (2), 39 (1935), 385–430.

Dixon, A. C.

(1) On a class of expansions in oscillating functions, *Proc. London Math. Soc.* (2), 3 (1905), 83–103.

(2) Harmonic expansions of functions of two variables, *Proc. London Math. Soc.* (2), 5 (1907), 411–78.

Dunham, J. L.

(1) The Wentzel-Brillouin-Kramers method of solving the wave-equation, *Phys. Review*, 41 (1932), 713–20.

(2) The energy levels of a rotating vibrator, *Phys. Review*, 41 (1932), 721–31.

ECKART, C.

(1) The penetration of a potential barrier by electrons, *Phys. Review*, 35 (1930), 1303–9.

HAAR, A.

(1) Zur Theorie der orthogonalen Funktionensysteme I, *Math. Annalen*, 69 (1910), 331–71.

(2) Zur Theorie der orthogonalen Funktionensysteme II, *Math. Annalen*, 71 (1911), 38–53.

HILB, E.

(1) Über Integraldarstellungen willkürlicher Funktionen, *Math. Annalen*, 66 (1909), 1–66.

(2) Über Reihenentwicklungen nach den Eigenfunktionen linearer Differentialgleichungen 2ter Ordnung, *Math. Annalen*, 71 (1911), 76–87.

HILBERT, D.

(1) Grundzügen einer allgemeinen Theorie der linearen Integralgleichungen, II, *Göttinger Nachrichten* (1904), 123–259.

HOBSON, E. W.

(1) On a general convergence theorem, and the theory of the representation of a function by series of normal functions, *Proc. London Math. Soc.* (2), 6 (1908), 349–95.

KEMBLE, E. C.

(1) Note on the Sturm-Liouville Eigenvalue-Eigenfunction problem with singular end-points, *Proc. Nat. Acad. Sci. U.S.A.* 19 (1933), 710–14.

(2) A contribution to the theory of the B.W.K. method, *Phys. Review*, 48 (1935), 549–61.

KNESER, A.

(1) Untersuchungen über die Darstellung willkürlicher Funktionen in der mathematischen Physik, *Math. Annalen*, 58 (1904), 81–147.

(2) Beiträge zur Theorie der Sturm-Liouvilleschen Darstellung willkürlicher Funktionen, *Math. Annalen*, 60 (1905), 402–23.

(3) Die Theorie der Integralgleichungen und die Darstellung willkürlicher Funktionen in der mathematischen Physik, *Math. Annalen*, 63 (1907), 477–524.

KRAMERS, H. A.

(1) Wellenmechanik und halbzahlige Quantisierung, *Zeitschrift für Physik*, 39 (1926), 828–40.

LANGER, R. E.

(1) On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order, *Trans. Amer. Math. Soc.* 33 (1931), 23–64.

(2) On the asymptotic solutions of differential equations, with an application to the Bessel functions of large complex order, *Trans. Amer. Math. Soc.* 34 (1932), 447–80.

(3) The asymptotic solutions of certain linear ordinary differential equations of the second order, *Trans. Amer. Math. Soc.* 36 (1934), 90–106.

(4) The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to the Stokes phenomenon, *Bull. Amer. Math. Soc.* 40 (1934), 545–82.

(5) On the asymptotic solutions of ordinary differential equations, with reference to the Stokes phenomenon about a singular point, *Trans. Amer. Math. Soc.* 37 (1935), 397–416.

(6) On the connection formulas and the solutions of the wave equation, *Phys. Review,* 51 (1937), 669–76.

LICHTENSTEIN, L.

(1) Zur Analysis der unendlichvielen Variablen. I. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung, *Rend. di Palermo,* 38 (1914), 113–66.

MILNE, W. E.

(1) On the degree of convergence of expansions in an infinite interval, *Trans. Amer. Math. Soc.* 31 (1929), 907–18.

(2) The numerical determination of characteristic numbers, *Phys. Review,* 35 (1930), 863–7.

PLANCHEREL, M.

(1) Integraldarstellungen willkürlicher Funktionen, *Math. Annalen,* 67 (1909), 519–34.

PRÜFER, H.

(1) Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen, *Math. Annalen,* 95 (1926), 499–518.

SCHRÖDINGER, E.

(1) Quantisierung als Eigenwertproblem, *Annalen der Physik* (4), 80 (1926), 437–90.

SCHUR, A.

(1) Zur Entwicklung willkürlicher Funktionen nach Lösungen von Systemen linearer Differentialgleichungen, *Math. Annalen,* 82 (1921), 213–39.

SONINE, N.

(1) Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries, *Math. Annalen,* 16 (1880), 1–80.

STONE, M. H.

(1) Certain integrals analogous to Fourier integrals, *Math. Zeitschrift,* 28 (1928), 654–76.

TAMARKIN, J.

(1) Sur quelques points de la théorie des équations différentielles, *Rend. di Palermo,* 34 (1912), 345–82.

(2) Some general problems of the theory of ordinary linear differential